Chapter 19  Exponential function

Question 1

Simplify \( (e^{-x})^2 e^{2x} \)

Ans:

\[
\left(e^{-x}\right)^2 e^{2x} = e^{-2x} e^{2x} = e^{-2x+2x} = e^0 = 1
\]

Question 2

A species of animal has population \( P(t) \) at time \( t \) given by

\[
P(t) = 10 - 5e^{-t}
\]

Sketch a graph of \( P(t) \) for \( 0 \leq t \leq 5 \).

What is the size of the population of the species as \( t \) becomes very large?

Ans:

From the graph, we see that the population starts at 5, but as \( t \) becomes very large, the population growth stagnates at a value of 10.

Question 3

Solve graphically the equation \( e^x + x = 0 \)

Ans:
Plotting the function \( y = e^x + x \) we get

The above plot intercepts the x-axis at approximately \(-0.6\), so \( x = -0.6 \) is the solution.

We can split up the functional \( y = e^x + x \) and plot \( y = e^x \) and \( y = -x \). Where the plots intercept is the solution.

We can see that the plots intercept with each other at an approximate value of \( x = -0.6 \).
Chapter 20  The logarithm function

Question 1
Write down the logarithm form of $6^5 = 7776$
Ans:
$\log_6 7776 = 5$

Question 2
Write down the logarithm form of $\left(\frac{1}{2}\right)^5 = 0.03125$
Ans:
$\log_{\frac{1}{2}} 0.03125 = 5$ or $\log_{0.5} 0.03125 = 5$

Question 3
Write the exponential form of $\log_7 2401 = 4$
Ans:
$7^4 = 2401$

Question 4
Given $\log 7 = 0.8451$ evaluate (a) $10^{0.8451}$ (b) $\log 700$ (c) $\log 0.07$ (d) $\log 49$
Ans:
As no base is specified, it is assumed to be 10.
(a) Taking the exponential form of $\log 7 = 0.8451$ we get
$7 = 10^{0.8451}$

(b) From (a), we can say that if $7 = 10^{0.8451}$, then
$700 = 100(7) = 10^2 \times 10^{0.8451} = 10^{2.8451}$
Taking the logarithm form we get
(c) From (a), we can say that if \( 7 = 10^{0.8451} \), then

\[
0.07 = \frac{7}{100} = 10^{-2} \times 10^{0.8451} = 10^{-1.1549}
\]

Taking the logarithm form we get

\[
\log 0.07 = -1.1549
\]

(d) From (a), we can say that if \( 7 = 10^{0.8451} \), then

\[
49 = 7^2 = (10^{0.8451})^2 = 10^{1.6902}
\]

Taking the logarithm form we get

\[
\log 49 = 1.6902
\]

**Question 5**

Evaluate \( \log_7 49 \)

**Ans:**

Using the following formula to change base

\[
\log_a X = \frac{\log_{10} X}{\log_{10} a} = \frac{\log X}{\log a}
\]

\[
\log_7 49 = \frac{\log 49}{\log 7} = \frac{1.6902}{0.8451} = 2
\]

**Question 6**

Simplify \( \log 7 + \log 36 \)

**Ans:**

Using the first law of logarithms,

\[
\log 7 + \log 36 = \log(7 \times 36) = \log 252
\]

**Question 7**

Simplify \( \log 1000 - \log 500 \)
Using the second law of logarithms,

\[ \log_{10}1000 - \log_{10}500 = \log_{10}\left(\frac{1000}{500}\right) = \log_{10}2 \]

Question 8
Simplify \( \log_{10}28 - \log_{10}12 + \log_{10}3 \)

Ans:

\[ \log_{10}28 - \log_{10}12 + \log_{10}3 = (\log_{10}28 - \log_{10}12) + \log_{10}3 = \log_{10}\left(\frac{28}{12}\right) + \log_{10}3 = \log_{10}\left(\frac{2}{3}\times3\right) = \log_{10}7 \]

Question 9
Simplify \( \log_{10}5y + \log_{10}4y - \log_{10}2y^2 \)

Ans:

\[ \log_{10}(5y) + \log_{10}(4y) - \log_{10}(2y^2) = \log_{10}(5y \times 4y) - \log_{10}(2y^2) = \log_{10}(20y^2) - \log_{10}(2y^2) = \log_{10}\left(\frac{20y^2}{2y^2}\right) = \log_{10}10 = 1 \]

Question 10
Simply \( 2\log_{10}5 + \log_{10}75 \)

Ans:

Using the third law of logarithms,

\[ 2\log_{10}5 + \log_{10}4 = \log_{10}5^2 + \log_{10}4 = \log_{10}(25 \times 4) = \log_{10}100 = 2 \]

Question 11
Express \( \log_{10}x + \ln x \)

Ans:

Firstly, we need to convert one term or the other so both terms are expressed to the same base.
\[
\log_{10} x + \ln x = \log x + \frac{\log x}{\log e} = \log x \left( 1 + \frac{1}{\log e} \right)
\]

Alternatively, \(\log_{10} x + \ln x = \frac{\ln x}{\ln 10} + \ln x = \ln x \left( 1 + \frac{1}{\ln 10} \right)\)

Question 12

Solve \(\log x + \ln x = 4\)

Ans:

Now \(\log x + \ln x = \log x \left( 1 + \frac{1}{\log e} \right) = 3.3026 \log x\)

So, \(3.3026 \log x = 4\)

Then \(\log x = \frac{4}{3.3026} = 1.2112\), so \(x = 10^{1.2112} = 16.2619\)

Chapter 26 Matrices

Question 1

Given \(A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \end{pmatrix}\) and \(B = \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix}\), find \(AB\) and \(BA\)

Ans:

\(A\) is a 2 x 3 matrix and \(B\) is a 2 x 2. Because the number of columns in \(A\) is not equal to the number of rows in \(B\), we cannot calculate \(AB\). However, we can calculate \(BA\) as the number of columns in \(B\) is equal to the number of rows in \(A\).

\[
BA = \begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 2 \times 4 & 1 \times 3 + 2 \times 0 & 1 \times (-1) + 2 \times 1 \\ 3 \times 2 + (-2) \times 4 & 3 \times 3 + (-2) \times 0 & 3 \times (-1) + (-2) \times 1 \end{pmatrix}
\]

Thus, \(BA = \begin{pmatrix} 10 & 3 & 1 \\ -2 & 9 & -5 \end{pmatrix}\)

Question 2

Use matrices to solve the following simultaneous equations.
2x + 7y = 4
3x - 2y = 31

Ans:

Expressing the simultaneous equations using matrices we have

\[
\begin{pmatrix} 2 & 7 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 31 \end{pmatrix}
\]

where \( A = \begin{pmatrix} 2 & 7 \\ 3 & -2 \end{pmatrix} \), \( X = \begin{pmatrix} x \\ y \end{pmatrix} \), and \( B = \begin{pmatrix} 4 \\ 31 \end{pmatrix} \)

If \( A^{-1} \) exists, then the solution is given by \( X = A^{-1}B \)

Now \( A^{-1} = \frac{1}{2(-2) - 7(3)} \begin{pmatrix} -2 & -7 \\ -3 & 2 \end{pmatrix} = \frac{1}{-25} \begin{pmatrix} -2 & -7 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{25} & \frac{7}{25} \\ \frac{3}{25} & -\frac{2}{25} \end{pmatrix} \)

\[
X = A^{-1}B = \begin{pmatrix} \frac{2}{25} & \frac{7}{25} \\ \frac{3}{25} & -\frac{2}{25} \end{pmatrix} \begin{pmatrix} 4 \\ 31 \end{pmatrix} = \begin{pmatrix} \frac{2}{25} \times 4 + \frac{7}{25} \times 31 \\ \frac{3}{25} \times 4 - \frac{2}{25} \times 31 \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}
\]

The solution occurs when \( x = 9 \) and \( y = -2 \)