Chapter 9 The Normal Distribution

Question 1

A probability distribution is given by the function \( h(z) = \frac{z^2}{30} \) for \( z = 0, 1, 2, 3, 4 \). Verify that the probability distribution is valid.

Ans: Firstly, for the probability distribution to be valid, the value of the function \( h(z) \) must not be negative over the values of \( z \) on the interval of 0 to 4. In this case, substituting any value of \( z \) into the function \( h(z) \) will always be positive. For example, if we substitute 2 for \( z \) in the function we get

\[
 h(z) = \frac{z^2}{30} = \frac{2^2}{30} = \frac{4}{30} = \frac{2}{15} > 0
\]

Secondly, the area under the function curve and between the intervals 0 and 4 must be less than or equal to 1. The area under the curve can be found using a graphical method (ie. plotting on graph paper and counting the approximate number of squares under the curve and arriving at the area). Alternatively, students familiar with integration can quickly arrive at an exact area under the curve being \( \frac{32}{45} = 0.71 \).

Thus the probability distribution is valid.

Question 2

If a probability distribution has a mean of 6 and a standard deviation of 2, what is the probability that a random variable will have a value greater than 9?

Ans: Firstly, we calculate the \( z \) score for \( x = 9 \) when \( \mu = 6 \) and \( \sigma = 2 \).

\[
 z = \frac{x - \mu}{\sigma} = \frac{9 - 6}{2} = \frac{3}{2} = 1.5
\]

Using a \( z \) score lookup table, the value when \( z = 1.5 \) is 0.4332. Now we must be careful here as this area represents the probability that the random variable will be between the mean of 6 and the value of 9. The probability of the random variable being below 6 is 0.5 and similarly, the total area and thus the probability of the random variable being above 6 is also 0.5.

The probability that the random variable will be above 9 is found by considering the total area above the mean then subtracting the area represented by the value 9 ie. \( 0.5 - 0.4332 = 0.0668 \).
Question 3

Using the above probability distribution in Question 2, what is the probability that the random variable will be exactly 5?

Ans: The probability is zero that a random variable will take on a particular value. Check for yourself and try to calculate the area under the curve between exactly 5 and exactly 5. It just can’t be done.

Question 4

Using the probability distribution in Question 2, what would be the probability that a random variable would take a value in the range of 6.5 and 7.5?

Ans: \[ z_{7.5} = \frac{7.5 - 6}{2} = \frac{1.5}{2} = 0.75 \] Using a lookup table 0.75 gives an area of 0.2734.

\[ z_{6.5} = \frac{6.5 - 6}{2} = \frac{0.5}{2} = 0.25 \] Using a lookup table, 0.25 gives an area of 0.0987

The probability that a random variable will take a value of between 6.5 and 7.5 is given by:

\[ \text{Area}_{z_{7.5}} - \text{Area}_{z_{6.5}} = 0.2734 - 0.0987 = 0.1747 \]

Question 5

Using the probably distribution in Question 2, what would be the probability that a random variable would take a value in the range of 5.5 and 7.5?

Ans: \[ z_{7.5} = \frac{7.5 - 6}{2} = \frac{1.5}{2} = 0.75 \] Using a lookup table 0.75 gives an area of 0.2734.

\[ z_{5.5} = \frac{5.5 - 6}{2} = \frac{-0.5}{2} = -0.25 \]

The reason we now have a negative value is that the random variable lower range value is less than the mean. The normal curve area table gives areas under the standard normal curve from the mean to higher values to the right of the mean. Because of symmetry around the mean, we can simply convert negative \( z \) score to a positive value by a change of sign, look up the area and simply add this area below the mean to the area above the mean.

The lookup table with the value 0.25 gives an area of 0.0987. Instead of subtracting like we did in Question 4, we now add the two areas together and get an area of 0.3721. This value represents the probability that the random variable will have a value between 5.5 and 7.5.
Question 6

Suppose a random variable occurring within a particular range had a probability of 0.6826. Calculate the range of values of the random variable if the area representing the probability was symmetrical around the mean.

Ans:

Because of symmetry, we simply divide the area by 2 and use the lookup table in reverse to what we have done in the previous questions. So, a half of 0.6826 is 0.3413. From the table, we get a $z$ score of exactly 1. Being symmetric around the mean, the lower $z$ score will have a value of -1.

Rearranging the formula $z = \frac{x - \mu}{\sigma}$, we have $x = z\sigma + \mu$

$x_{\text{upper}} = z\sigma + \mu = 1 \times 2 + 6 = 8$

$x_{\text{lower}} = z\sigma + \mu = -1 \times 2 + 6 = 4$

Thus the range of values is 4 to 8.