Question 1 (COMPULSORY QUESTION)

Evaluate each of the following showing all work, or if tables are used, indicating the table entry concerned:

(a) \( \int x(x - 4)(x + 7) \, dx \)

(b) \( \int_{2}^{3} \frac{t^6 - t^2}{t^4} \, dt \)

(c) \( \int_{0}^{\frac{\pi}{3}} \frac{\sin \theta}{\cos^2 \theta} \, d\theta \)

(d) \( \int \frac{6}{x^2 - 2x - 8} \, dx \)

(e) \( \int x^3 \ln(x) \, dx \)

(f) \( \int_{0}^{2} \frac{dx}{4x - 5} \) (After evaluation write if convergent or divergent).

Question 2 overleaf
ANSWER ANY FOUR FROM THE NEXT SEVEN QUESTIONS

Question 2

(a) Use the Trapezoidal Rule to evaluate the integral \( \int_{0}^{3} e^{x^2} \, dx, \ n = 5 \) and construct an error bound for your estimate.

(b) Sketch the region bounded by the graphs \( x + y^2 = 4 \) and \( x + y = 2 \) and find the area of this enclosed region by means of a definite integral.

Question 3

(a) A commercial dirigible used for outdoor advertising has a helium-filled balloon in the shape of an ellipse revolved about its major axis. If the balloon is 41.3m long and 12m in diameter, what volume of helium is required to fill it. (Neglect wall thickness). The equation of the ellipse is

\[
\left( \frac{x}{20.65} \right)^2 + \left( \frac{y}{6} \right)^2 = 1.
\]

Question 3 continued overleaf
(b) The curve described by the cable of the suspension bridge shown in the figure below is given by \( y = h \left( \frac{x}{\ell} - 1 \right)^2 \) where \( x \) is the distance measured from one end of the bridge and \( 2\ell \) is the total length of the bridge. What is the length of the cable?

![Diagram of a suspension bridge with a cable](image)

**Question 4**

(a) Find the derivative, \( df(x)/dx \), where

\[
 f(x) = \int_{\tan^{-1}x}^{\infty} \frac{1}{\sqrt{1+t^4}} \, dt
\]

(b) A floodgate is in the shape of an isosceles trapezoid. Find the location of the centroid of the floodgate if the upper base is 20m, the lower base is 12m, and the height between the bases is 6.0m.

![Diagram of an isosceles trapezoid](image)
Question 5

(a) Water flows from a vertical cylindrical storage tank through a circular pipe at the bottom. If the outflow velocity $U_b$ is given by Torricelli’s Equation, $U_b = \sqrt{2gh}$, how long will it take for the tank to empty given that depth of water, $h=70\text{cm}$ and diameters are: $d_1 = 500\text{cm}$, $d_2 = 1\text{cm}$? (Take $g = 981\text{cm/sec}^2$).

(b) Suppose the ANZ Bank gives you a home loan of $100,000.00 at an annual interest rate of 7 percent to build a house. If the loan is compounded daily, what will be your total loan after a month if you do not make any repayments during that time?

Question 6 overleaf
Question 6

(a) Find a power series representation for the following function and determine the interval of convergence

\[ f(x) = \frac{1 + x^2}{1 - x^2} \]

(b) If an electric discharge is passed through hydrogen gas, a spectrum of isolated parallel lines, called the Balmer Series, is formed. The wavelengths \( \lambda \) (in nm) of the light for these lines is given by the formula

\[ \frac{1}{\lambda} = 1.097 \times 10^{-7} \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \quad n = 3,4,5,... \]

Find the wavelengths of the first three lines and the shortest wavelength of all the lines of the series.

Question 7

(a) Find the Maclaurin Series for \( f(x) \), where

\[ f(x) = \frac{1}{1 + x} \]

(b) Integrate the result in (a) to find the Maclaurin Series for \( f(x) = \ln(1 + x) \).

(c) Use the result in part (b) to find a Maclaurin Series for \( \ln(1 - x) \)

(d) Use the results of parts (b) and (c) to find Maclaurin Series

\[ \ln \left( \frac{1 + x}{1 - x} \right) \]

Question 8 overleaf
Question 8

(a) A vertical dam has a triangular gate as shown below. Find the hydrostatic force against the gate. [Use density of water, \( \rho = 1000 \text{Kg/m}^3 \) and \( g = 9.8 \text{m/sec}^2 \)].

(b) After 5 days a sample of radon-222 decayed to 68% of its original amount.

(i) What is the half-life of radon-222?

(ii) How long would it take the sample to decay to 10% of its original amount?
SOLUTIONS TO WINTER EXAMINATION - 2001

QUESTION 1

(a) The indefinite integral is evaluated using the power rule:

\[ \int x(x - 4)(x + 7)dx = \int (x^2 - 4x + 7x - 28)dx = \int (x^3 - 4x^2 + 7x^2 - 28x)dx \]
\[ = \int (x^3 + 3x^2 - 28x)dx = \frac{x^4}{4} + \frac{3x^3}{3} - 28 \frac{x^2}{2} + C \]
\[ = \frac{1}{4}x^4 + x^3 - 14x^2 + C, \]
where \( C \) is an arbitrary constant of integration.

(b) The indefinite integral is integrated as follows:

\[ \int_2^3 \frac{t^6 - t^2}{t^4} \, dt = \int_2^3 \left( t^2 - \frac{1}{t^2} \right) \, dt \]
\[ = \left[ \frac{t^3}{3} - \frac{t^{-1}}{-1} \right]_2^3 = \left[ \frac{t^3}{3} + \frac{1}{t} \right]_2^3 \]
\[ = \left[ \frac{3^3}{3} + \frac{1}{3} - \frac{2^3}{3} - \frac{1}{2} \right] = \left[ 9 + \frac{1}{3} - \frac{8}{3} - \frac{1}{2} \right] \]
\[ = \frac{54 + 2 - 16 - 3}{6} = \frac{37}{6}. \]

(c) We first evaluate the indefinite integral by substitution.

\[ \int \frac{\sin(\theta)}{\cos^2(\theta)} \, d\theta = \int \frac{d(-\cos(\theta))}{\cos^2(\theta)} \quad \text{since} \quad d(-\cos(\theta)) = \sin(\theta) \, d\theta \]
\[ = -\int \frac{1}{u^2} \, du \quad \text{where} \quad u = \cos(\theta) \]
\[ = \frac{1}{u} + C = \frac{1}{\cos(\theta)} + C, \]
where \( C \) is an arbitrary constant of integration. The definite integral is now easily evaluated as:

\[ \int_0^{\pi/3} \frac{\sin(\theta)}{\cos^2(\theta)} \, d\theta = \left[ \frac{1}{\cos(\theta)} \right]_0^{\pi/3} = \left[ \frac{1}{1/2} - \frac{1}{1} \right] = 1. \]

(d) This indefinite integral is evaluated using the method of partial fractions. The quadratic in the denominator of the integrand

\[ x^2 - 2x - 8 = (x - 4)(x + 2). \]
By the method of partial fractions there exists constants $A$ and $B$ such that
\[
\frac{6}{(x - 4)(x + 2)} = \frac{A}{x - 4} + \frac{B}{x + 2}.
\]

We use the method of undetermined coefficients to find the constants. Rewriting the right hand side of this equation on a common denominator we have
\[
\frac{6}{(x - 4)(x + 2)} = \frac{A(x + 2) + B(x - 4)}{(x - 4)(x + 2)} = \frac{(A + B)x + (2A - 4B)}{(x - 4)(x + 2)}.
\]

Equating coefficients of the polynomial in the numerator of both sides, constants $A$ and $B$ must satisfy the two equations
\[
A + B = 0, \quad 2A - 4B = 6.
\]

Hence $B = -1$ and $A = 1$, and
\[
\frac{6}{(x - 4)(x + 2)} = \frac{1}{x - 4} - \frac{1}{x + 2}.
\]

The integral can now be evaluated
\[
\int \frac{6}{(x - 4)(x + 2)} \, dx = \int \left[ \frac{1}{x - 4} - \frac{1}{x + 2} \right] \, dx = \int \frac{1}{x - 4} \, dx - \int \frac{1}{x + 2} \, dx = \ln |x - 4| - \ln |x + 2| + C = \ln \left| \frac{x - 4}{x + 2} \right| + C,
\]
where $C$ is an arbitrary constant of integration.

(e) The integral is integrated using the method of integration by parts.
\[
\int x^2 \ln(x) \, dx = \int \ln(x) \, d\left(\frac{x^3}{3}\right)
\]
\[
= \ln(x) \left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right) \, d(\ln(x))
\]
\[
= \ln(x) \left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right) \left(\frac{1}{x}\right) \, dx = \ln(x) \left(\frac{x^3}{3}\right) - \int \left(\frac{x^2}{3}\right) \, dx
\]
\[
= \ln(x) \left(\frac{x^3}{3}\right) - \frac{x^3}{9} + C = \frac{x^3}{3} \left(\ln(x) - \frac{1}{3}\right) + C,
\]
where $C$ is an arbitrary constant of integration.

(f) This integral is an improper integral because the integrand is undefined at $x = 5/4$. Hence by definition
\[
\int_0^2 \frac{1}{4x - 5} \, dx = \lim_{t \to 5/4^-} \int_0^t \frac{1}{4x - 5} \, dx + \lim_{t \to 5/4^+} \int_t^2 \frac{1}{4x - 5} \, dx.
\]

Note that each of these integrals on the right hand side must be convergent for the integral to be convergent. Now the indefinite integral
\[
\int \frac{1}{4x - 5} \, dx = \frac{1}{4} \int \frac{1}{u} \, du \quad \text{where} \quad u = 4x - 5
\]
\[
= \frac{1}{4} \ln|u| + C = \frac{1}{4} \ln|4x - 5| + C,
\]

2
where $C$ is an arbitrary constant of integration.

Hence
\[
\lim_{t \to 5/4^-} \int_0^t \frac{1}{4x - 5} \, dx = \lim_{t \to 5/4^-} \left[ \ln|4x - 5| \right]_0^t = \lim_{t \to 5/4^-} \left[ \frac{\ln|4t - 5|}{4} - \frac{\ln(5)}{4} \right] = -\infty,
\]
and
\[
\lim_{t \to 5/4^+} \int_0^t \frac{1}{4x - 5} \, dx = \lim_{t \to 5/4^+} \left[ \ln|4x - 5| \right]_0^t = \lim_{t \to 5/4^+} \left[ \frac{\ln(3)}{4} - \frac{\ln|4t - 5|}{4} \right] = +\infty.
\]
As both integrals are divergent, then the given proper integral is divergent.

**QUESTION 2**

For the Trapezoidal Rule with $n = 5$ and interval $[1, 3]$, $\Delta x = (3 - 1)/5 = 2/5 = 0.4$. Hence
\[
\int_1^3 f(x) \, dx = \int_1^3 \frac{e^x}{x} \, dx = \frac{\Delta x}{2} \left[ f(1) + 2f(1.4) + 2f(1.8) + 2f(2.2) + 2f(2.6) + f(3) \right]
\]
\[
= 0.2 \left[ \frac{1}{1} + 2^{1.4} + 2^{1.8} + 2^{2.2} + 2^{2.6} + \frac{3}{3} \right]
\]
\[
\approx 0.2(2.718 + 5.793 + 6.722 + 8.205 + 10.357 + 6.695) = 8.10.
\]
To calculate the error bound, we first find $K$ where
\[
K = \max \{ f''(x) : x \in [1, 3] \}.
\]
Now
\[
f(x) = \frac{e^x}{x}, \quad f'(x) = \frac{e^x}{x^2}, \quad \text{and}
\]
\[
f''(x) = \frac{e^x}{x^3} - \frac{2e^x}{x^2} + \frac{6e^x}{x^3} = \frac{(x^3 - 3x^2 + 6x - 6)e^x}{x^3}.
\]
It is possible to show that the biggest value of the $f''(x)$ occurs at $x = 3$. This can be done by calculating the third derivative
\[
f'''(x) = \frac{e^x}{x^4} - \frac{3e^x}{x^3} + \frac{6e^x}{x^4} - \frac{6e^x}{x^4} = \frac{(x^3 - 3x^2 + 6x - 6)e^x}{x^4}.
\]
Now let us examine the cubic polynomial $g(x) = x^3 - 3x^2 + 6x - 6$. Its derivative $g'(x) = 3x^2 - 6x + 6$ and is always positive, since $g'(x)$ does not equal zero for any real $x$. This means the cubic is a strictly increasing function and can only cut the $x$-axis at one point. This point $x^*$ lies in the interval $[1, 2]$, since $g(1) = -2$ and $g(2) = 2$. Hence to the left of $x^*$, $f'''(x)$ is negative, and to the right of $x^*$, $f'''(x)$ is positive. In terms of $f''$ this means that $f''$ is decreasing on the left of $x^*$ and increasing on the right of $x^*$. As the function $f''$ is continuous on $[1, 3]$ it must attain its maximum at either $x = 1$ or $x = 3$. Now
\[
f''(1) = e \quad \text{and} \quad f''(3) = \frac{5e^3}{27}.
\]
and we may take $K = f''(3)$ since it is greater than the value $f''(1)$. For the integral
\[
\int_a^b f(x) \, dx,
\]
the error bound is given by the formula
\[ E = \frac{K(b - a)^3}{12n^2}. \]
In this example then, the error bound is
\[ E = \frac{K(b - a)^3}{12n^2} = \frac{5e^3(2)^3}{(27)(12)(5)^2} \approx 0.099. \]

(b) At points of intersection, both equations must simultaneously be satisfied, that is,
\[ x + y^2 = 4, \quad \text{and} \quad x + y = 2. \]
Solving these equations, substituting for \( x = 2 - y \) into the first equation, we find
\[ y^2 - y - 2 = (y + 1)(y - 2) = 0, \]
which gives that \( y = -1 \) or \( y = 2 \). The points of intersection are therefore \((3, -1)\) and \((0, 2)\).

The region whose area is to be found is given in Figure 1 Using horizontal strips as shown in the figure, the left end coordinate \( x_L = 2 - y \) and the right end coordinate is \( x_R = 4 - y^2 \). The range of integration on the \( y \)-axis is from \( y = -1 \) to \( y = 2 \). Hence the required area is
\[
A = \int_{-1}^{2} \left[ (4 - y^2) - (2 - y) \right] dy = \int_{-1}^{2} 2 + y - y^2 dy
\]
\[
= \left[ 2y + \frac{y^2}{2} - \frac{y^3}{3} \right]_{-1}^{2} = \left[ 2(2) + \frac{2^2}{2} - \frac{2^3}{3} \right] - \left[ 2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3} \right]
\]
\[
= \left[ 4 + 2 - \frac{8}{3} \right] - \left[ -2 + \frac{1}{2} + \frac{1}{3} \right] = 4.5
\]

**QUESTION 3**

(a) Taking vertical strips of thickness \( dx \) and rotating about the \( x \)-axis, the area of the circle disk obtained is \( \pi y^2 \). Its elemental volume \( dV = (\text{Area})(\text{thickness}) = \pi y^2 dx \). The equation of the ellipse
\[
\left( \frac{y}{6} \right)^2 = 1 - \left( \frac{x}{20.65} \right)^2.
\]
enables us to find $y^2$ explicitly as a function of $x$, that is

$$y^2 = 36 \left[ 1 - \left( \frac{x}{20.65} \right)^2 \right].$$

Hence “summing up” all the elemental volumes for $x$ from $-20.65$ to $20.65$, we obtain the volume $V$ of the dirigible, see Figure 2 as:

$$V = \int_{-20.65}^{20.65} \pi y^2 \, dx = 2 \int_{0}^{20.65} \pi y^2 \, dx \quad \text{due to even symmetry about y-axis}$$

$$= 72\pi \int_{0}^{20.65} \left[ 1 - \left( \frac{x}{20.65} \right)^2 \right] \, dx$$

$$= 72\pi \left[ x - \frac{x^3}{3(20.65)^2} \right]_{0}^{20.65} = 72\pi \left[ 20.65 - \frac{20.65^3}{3} \right]$$

$$= 48(20.65)\pi = 991.2\pi \approx 3113.95 \text{ m}^3.$$

Figure 2: Figure of dirigible for Question 3(a).

(b) We consider the suspension bridge in Figure 3. The equation of the curve is given by

$$y = f(x) = h \left( \frac{x}{\ell} - 1 \right)^2.$$
with $x \in [0, 2\ell]$. Using the chain rule the function’s derivative is

$$\frac{dy}{dx} = f'(x) = \frac{2h}{\ell} \left( \frac{x}{\ell} - 1 \right),$$

and

$$\left( \frac{dy}{dx} \right)^2 = \frac{4h^2}{\ell^2} \left( \frac{x}{\ell} - 1 \right)^2.$$

The length $L$ of the suspension bridge is given by the arc length formula

$$L = \int_0^{2\ell} \left( 1 + \left( \frac{dy}{dx} \right)^2 \right)^{1/2} dx = \int_0^{2\ell} \left( 1 + \frac{4h^2}{\ell^2} \left( \frac{x}{\ell} - 1 \right)^2 \right)^{1/2} dx.$$

To simplify the integral we first make the substitution

$$u = \frac{x}{\ell} - 1, \quad \text{with} \quad du = \frac{dx}{\ell},$$

and limits $u = -1$ when $x = 0$ and $u = 1$ when $x = 2\ell$. Then the definite integral becomes

$$L = \int_{-1}^{1} \left( 1 + \frac{4h^2}{\ell^2} u^2 \right)^{1/2} (\ell du) = 2h \int_{-1}^{1} \left( \frac{\ell^2}{(2h)^2} + u^2 \right)^{1/2} \frac{du}{\ell} = 2h \int_{-1}^{1} (a^2 + u^2)^{1/2} du,$$

where $a = \ell/(2h)$, to simplify the integrand.

Using the formula table we find that

$$\int (a^2 + u^2)^{1/2} du = \frac{u}{2} \left( a^2 + u^2 \right)^{1/2} + \frac{a^2}{2} \ln \left| u + (a^2 + u^2)^{1/2} \right|.$$

Hence the length can be evaluated as

$$L = 2h \int_{-1}^{1} (a^2 + u^2)^{1/2} du \quad \begin{array}{c} = \left[ \frac{u}{2} \left( a^2 + u^2 \right)^{1/2} + \frac{a^2}{2} \ln \left| u + (a^2 + u^2)^{1/2} \right| \right]_{-1}^{1} \\ = h \left[ 2 \left( 1 + a^2 \right)^{1/2} + a^2 \ln \left( \frac{(1 + a^2)^{1/2} + 1}{(1 + a^2)^{1/2} - 1} \right) \right], \end{array}$$

where $a = \ell/(2h)$.

**QUESTION 4**

(a)

We are to find the derivative of $f$ where function $f$ is defined by the integral

$$f(x) = \int_{\tan(x)}^{x^2} \frac{1}{\sqrt{1 + t^4}} dt.$$
which has variable upper and lower limits. The fundamental theorem of calculus will be used. First we put the integral in a form for which this theorem can be used. We write

\[ f(x) = \int_{\tan(x)}^{x^2} \frac{1}{\sqrt{1 + t^4}} \, dt \]

\[ = \int_{a}^{\tan(x)} \frac{1}{\sqrt{1 + t^4}} \, dt + \int_{a}^{x^2} \frac{1}{\sqrt{1 + t^4}} \, dt \]

\[ = -\int_{a}^{\tan(x)} \frac{1}{\sqrt{1 + t^4}} \, dt + \int_{a}^{x^2} \frac{1}{\sqrt{1 + t^4}} \, dt, \]

where \( a \) is any constant in the range of integration. We note that the integrand is a continuous function for all real values of the variable \( t \). Now the fundamental theorem of calculus states that given function \( G \) defined by

\[ G(x) = \int_{a}^{x} g(t) \, dt, \quad \text{then} \quad G'(x) = g(x). \]

We consider each integral to be differentiated term by term. Write the first integral as

\[ H(x) = \int_{a}^{\tan(x)} \frac{1}{\sqrt{1 + t^4}} \, dt = \int_{a}^{v} \frac{1}{\sqrt{1 + t^4}} \, dt, \]

where \( v = V(x) = \tan(x) \). Then \( H \) is written as the composition of two functions, namely:

\[ H(x) = (K \circ V)(x) = K(V(x)) \quad \text{where} \quad V(x) = \tan(x), \]

and where

\[ K(v) = \int_{a}^{v} \frac{1}{\sqrt{1 + t^4}} \, dt, \]

By the chain rule and the fundamental of calculus,

\[ H'(x) = K'(v)V'(x) \quad \text{where} \quad v = V(x) = \tan(x) \]

\[ = \left[ \frac{1}{\sqrt{1 + v^4}} \right] \sec^2(x) \quad \text{where} \quad v = V(x) = \tan(x) \]

\[ = \frac{\sec^2(x)}{\sqrt{1 + \tan^4(x)}} \]

Similarly we can evaluate the derivative of the second integral. Formally we write

\[ L(x) = (M \circ W)(x) = M(W(x)) \quad \text{where} \quad w = W(x) = x^2, \]

and where

\[ M(w) = \int_{a}^{w} \frac{1}{\sqrt{1 + t^4}} \, dt, \]

By the chain rule and the fundamental of calculus,

\[ L'(x) = M'(w)W'(x) \quad \text{where} \quad w = W(x) = x^2 \]

\[ = \left[ \frac{1}{\sqrt{1 + w^4}} \right] 2x \quad \text{where} \quad w = W(x) = x^2 \]

\[ = \frac{2x}{\sqrt{1 + x^4}}. \]
Hence the derivative of the given function is
\[ f'(x) = -\frac{\sec^2(x)}{\sqrt{1 + \tan^2(x)}} + \frac{2x}{\sqrt{1 + x^2}} \]

(b) From symmetry, the centroid has to be on the vertical line half-way from the edges. Assuming this to be the \( y \)-axis and that the origin is at the base of the gate, Point \( A \) has coordinates \((6,0)\) and point \( B \) has coordinates \((10,6)\). The flood gate is shown in Figure 4. Consider a small element of area at height \( y \) above the \( x \)-axis, depicted in the figure, with width \( dy \), length \( 2x \) and elemental area \( dA = 2xdy \). The \( y \) coordinate of the centroid of the floodgate is given by formula to be
\[ \bar{y} = \frac{\sum dAy}{\sum dA}, \]
with summation being over all such elemental areas. Taking this approximation to the limit by letting the number of such strips tend to infinity we obtain the formula for
\[ \bar{y} = \frac{\int_0^6 2xydy}{A}, \]
where \( A \) is the total area of the floodgate. Now the total area is that of a trapezium and can be easily calculated as
\[ A = \frac{1}{2}(12 + 20)(6) = 96(\text{units}). \]
To evaluate the integral we find the equation of the straight line connecting points \( A \) and \( B \). It is found to be given by the formula
\[ y - 0 = \left( \frac{6 - 0}{10 - 6} \right)(x - 6), \quad \text{or} \quad x = 6 + \frac{2}{3}y. \]
Using this formula we substitute for \( x \) and integrate with respect to \( y \) to evaluate
\[ \int_0^6 2xydy = \int_0^6 \left(12 + \frac{4}{3}y\right)ydy = \int_0^6 \left[12y + \frac{4}{3}y^2\right]dy = \left[6y^2 + \frac{4}{9}y^3\right]_0^6 = 216 \left[\frac{13}{9}\right]. \]
Hence the \( y \) coordinate of the centroid \( \bar{y} \), has value
\[ \bar{y} = \frac{(216)(13)}{(96)(9)} \approx 3.25(\text{metres}), \]
from the base of the gate. The centroid relative to the defined coordinate structure has coordinates
\[ \bar{x} = 0 \text{(m)} \quad \text{and} \quad \bar{y} = 3.25 \text{(m)}. \]

**QUESTION 5**

(a) We are given that water flows from a cylindrical storage tank through a circular pipe at the bottom, see Figure 5. Let \( V \) be the volume of water in the tank at any time and \( h \) be the depth of water in the tank at any time \( t \). Suppose that in a small interval \( dt \) the water height decreases by an amount \( dh \). Then the decrease in volume \( dV \) over this interval equals the amount of water that has left the tank. Given the dimensions shown, then the decrease in volume is
\[ dV = - (\text{Area of tank base})dh = - \left( \frac{\pi}{4} d_1^2 \right) dh, \]
and the volume outflow is
\[ dV = (\text{Cross sectional area of pipe})(\text{Outflow speed})dt = \left( \frac{\pi}{4} d_2^2 \right) U_B dt. \]
Hence we conclude that
\[ - \left( \frac{\pi}{4} d_1^2 \right) dh = \left( \frac{\pi}{4} d_2^2 \right) U_B dt, \]
or
\[ -(500)^2 dh = \sqrt{2gh} dt. \]
We now integrate noting that \( h \) decrease from 70 cm to 0 cm in time interval \( T \), to obtain
\[ - \frac{500^2}{\sqrt{2g}} \int_{70}^{0} \frac{1}{h^{1/2}} dh = \int_{0}^{T} dt, \]
\[ - \frac{500^2}{\sqrt{2g}} \left[ \frac{h^{1/2}}{(1/2)} \right]_{70}^{0} = T. \]
Rearranging terms we find the time $T$ taken to be

$$T = \frac{500^2}{\sqrt{2g}} \sqrt{70} \approx 9443 \text{ (secs)} \approx 26 \text{ (hrs)} 14 \text{ (min)}.$$

(b) Let $A_0 = $100,000.00 be the initial amount of the loan and $A(t)$ be the amount of the loan at any time $t$ thereafter. We assume $t$ is measured in days. The annual interest rate $r = 0.07$ and if compounded daily we set $n = 365$. Assuming that there are 30 days in a month, the total amount of the loan repayment after 1 month is given by the formula

$$A(30) = A_0 \left(1 + \frac{r}{n}\right)^{30} \approx 100000(1.0001918)^{30} \approx 100000(1.0057694) = 100576.94 \text{ (dollars)}.$$

**QUESTION 6**

(a) We are given the function $f$ defined by

$$f(x) = \frac{1 + x^2}{1 - x^2} = 1 + \frac{2x^2}{1 - x^2}.$$ 

It is assumed that we find the Maclaurin series,

$$M = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k,$$

that is, the power series of $f$ about $x = 0$.

To do this requires calculation of the derivative coefficients in the expansion. A quick check shows this is very tedious once the first derivative is calculated. A simple way is required. We use the expansion

$$\frac{1}{1-w} = 1 + w + w^2 + \cdots + w^n + \cdots = \sum_{k=0}^{\infty} w^k,$$

which is convergent for all $|w| < 1$ and divergent for all other values of $w$. The can be simply verified by long division or by direct calculation of the Maclaurin series coefficients.

Setting $w = x^2$ gives us the power series expansion

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k},$$

which is convergent for $|x^2| < 1$, or $|x| < 1$. Now using the expression for $f$ we can write

$$f(x) = 1 + 2x^2 \sum_{k=0}^{\infty} x^{2k} = 1 + \sum_{k=0}^{\infty} 2x^{2(k+1)} = 1 + \sum_{m=1}^{\infty} 2x^{2m},$$

having set the summation index to be $m = k + 1$. This expression gives the power series expansion for $f$ about $x = 0$ which is convergent for $|x| < 1$.

(b) The wavelength $\lambda(nm)$ of the light for the Balmer Series spectrum of parallel lines, is given by the formula

$$\frac{1}{\lambda_n} = (1.097)10^{-2} \left(\frac{1}{2^2} - \frac{1}{n^2}\right) \quad n = 3, 4, 5, \ldots$$
The wavelengths of the first three lines are simply calculated at \( n = 3, n = 4 \) and \( n = 5 \). They are
\[
\frac{1}{\lambda_3} = (1.097)10^{-2} \left( \frac{1}{2^2} - \frac{1}{3^2} \right) = 0.01097 \left( \frac{5}{36} \right) \quad \text{which gives} \quad \lambda_3 \approx 656.34(\text{nm}).
\]
\[
\frac{1}{\lambda_4} = (1.097)10^{-2} \left( \frac{1}{2^2} - \frac{1}{4^2} \right) = 0.01097 \left( \frac{3}{16} \right) \quad \text{which gives} \quad \lambda_4 \approx 486.17(\text{nm}).
\]
\[
\frac{1}{\lambda_5} = (1.097)10^{-2} \left( \frac{1}{2^2} - \frac{1}{5^2} \right) = 0.01097 \left( \frac{21}{100} \right) \quad \text{which gives} \quad \lambda_5 \approx 434.08(\text{nm}).
\]

In the sequence of numbers \( \lambda_n \) the one of shortest wavelength will be given by that number in the sequence \( w_n = \frac{1}{\lambda_n} = (1.097)10^{-2} \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \quad n = 3, 4, 5, \ldots \)
that has the largest wavelength. Since
\[
\lim_{n \to \infty} \frac{1}{n^2} = 0,
\]
we see that the sequence \( w_n \) is an increasing sequence and has as its limit (using the relevant limit theorems)
\[
\lim_{n \to \infty} w_n = \lim_{n \to \infty} (1.097)10^{-2} \left( \frac{1}{2^2} - \frac{1}{n^2} \right) = (1.097)10^{-2} \frac{1}{2^2} = \frac{0.02097}{4},
\]
which is its least upper bound. We note then that the shortest wavelength of all lines is unattainable for a finite value of \( n \), but the wavelengths of the lines decrease tending to a limit wavelength of
\[
\lambda = \frac{4}{0.01097} \approx 364.63(\text{nm}).
\]

**QUESTION 7**

(a) We are given the function \( f \) defined by
\[
f(x) = \frac{1}{1 + x}.
\]
We first find a general formula for the derivatives of \( f \). Now
\[
f'(x) = (-1)(1 + x)^{-2}, \quad f''(x) = (-1)^2(2)(1 + x)^{-3}, \quad f'''(x) = (-1)^3(3)(2)(1 + x)^{-4},
\]
\[
f^4(x) = (-1)^4(4)(3)(2)(1 + x)^{-5} \quad \cdots \quad f^k(x) = (-1)^k k!(1 + x)^{-k+1} \cdots .
\]
Hence the Maclaurin series for \( f \) is
\[
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{(-1)^k k!(1)^{-k+1}}{k!} x^k = \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots .
\]
The same result could be obtained by polynomial long division or by setting \( w = -x \) in Question 6 (a). The series is clearly convergent to the value \( f(x) \) for all \( x \) satisfying \( |x| < 1 \).
(b) Integrating the Maclaurin series term by term in its region of convergence we obtain.

\[\ell n(1 + x) = \int \frac{1}{1+x} dx = \sum_{k=0}^{\infty} (-1)^k \int x^k dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.\]

Again this series converges for those \(x\) satisfying \(|x| < 1\).

(c) The answer is simply obtained by setting \(x\) to \(-x\). We obtain:

\[\ell n(1 - x) = \sum_{k=0}^{\infty} (-1)^k \frac{(-x)^{k+1}}{k+1} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots.\]

which converges for all \(x\) with \(|x| < 1\).

(d) The answer is easily obtained since

\[\ell n \left(\frac{1+x}{1-x}\right) = \ell n(1 + x) - \ell n(1 - x).\]

Expanding both terms on the right handside for all \(x\) satisfying \(|x| < 1\), we obtain the following expansion

\[\ell n \left(\frac{1+x}{1-x}\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)\]

QUESTION 8

(a) The vertical dam with triangular gate is shown in Figure 6. At any depth \(h\) from the surface and in any direction, pressure \(dP\) is \(\rho gh\). This pressure is uniform along a horizontal strip at depth \(h\) below the water level with thickness \(dh\). Using the triangles in Figure 7. we determine that the lengths \(L\) and \(x\) pictured are

\[L = 6 \tan(\pi/6) \approx 3.464, \quad \text{and} \quad x = (6 - h) \tan(\pi/6).\]
The area of the strip $dA$ is then
\[ dA = 2Ldh = 2(6 - h)\tan(\pi/6)dh. \]
The element of force $dF$ on this strip
\[ dF = dPdA = (\rho gh)(2(6 - h)\tan(\pi/6)dh). \]
The total hydrostatic force $F$ on the gate is then accumulated as

![Figure 7: Figure for triangles in Question 8(a).](image)

\[ F = \int_0^6 2\rho g\tan(\pi/6)(6h - h^2)dh \]
\[ = 2(1000)(9.8) \frac{1}{\sqrt{3}} \left[ 3h^2 - \frac{h^3}{3} \right]_0^6 \]
\[ \approx 407378 \text{(Newtons)} \]

(b) (i) We are given that after 5 days a sample of radon-222 decayed to 68\% of its original amount. Let the amount of radon at any time $t \geq 0$, be given by $A(t)$ and the amount at initial time be $A_0$. Time is assumed measured in days.

We know from the equations of radioactive decay that
\[ A(t) = A_0 \exp(-kt), \]
where $k$ is the positive decay constant. Now in 5 days we have that 68\% of the original amount remains, so
\[ 0.68A_0 = A_0 \exp(-5k), \]
which allows us to find the unknown decay constant. Simplifying this expression we find that
\[ k = -\frac{\ln(0.68)}{5} \approx 0.0771324. \]
The time $T_{1/2}$ defining the half life of radon, is that time taken to reduce the original amount by half, so $T_{1/2}$ is given by

$$0.5 = \exp(-kT_{1/2}),$$

from which we obtain

$$T_{1/2} = -\frac{\ln(0.5)}{0.0771324} \approx 8.99\text{days}.$$

So the half life of radon may be taken to be around 9 days.

(ii) With the information given above the amount of radon remaining after a period of time $t$ is given by

$$A(t) = A_0 \exp(-kt),$$

where $k \approx 0.0771324$. The time $T$ taken for radon to decay to 10\% of the original amount $A_0$ is given by

$$0.1 = \exp(-0.0771324T).$$

Hence the required time is

$$T = -\frac{\ln(0.1)}{0.0771324} \approx 29.85(\text{days}).$$