QUESTION 1

(a) (i) The definite integral is evaluated as follows recognising that:

\[
|t^2 - 1| = \begin{cases} 
  t^2 - 1 & \text{if } t \geq 1 \\
  -t^2 + 1 & \text{if } 0 \leq t < 1 
\end{cases}
\]

If \(0 \leq x < 1\)

\[
\int_0^x |t^2 - 1| \, dt = \int_0^x (-t^2 + 1) \, dt = \left[ -\frac{t^3}{3} + t \right]_0^x = -\frac{x^3}{3} + x.
\]

If \(1 \leq x\)

\[
\int_0^x |t^2 - 1| \, dt = \int_0^x (t^2 - 1) \, dt = \left[ \frac{t^3}{3} - t \right]_0^x = x^3 - x.
\]

(ii) Error in the question, the limits of integration should have been from \(\pi/8\) to \(\pi/2\). We first evaluate the indefinite integral. Note that \(\sec^2(2x) = 1 + \tan^2(2x)\) and \(d(\tan(2x)) = 2 \sec^2 (2x) \, dx\).

\[
\int \tan^2(2x) \sec^4(2x) \, dx = \frac{1}{2} \int \tan^2(2x) \left(1 + \tan^2(2x)\right) d(\tan(2x))
\]

\[
= \frac{1}{2} \int t^2 (1 + t^2) \, dt \quad \text{where } t = \tan(2x)
\]

\[
= \frac{1}{2} \int t^2 + t^4 \, dt = \frac{1}{2} \left( \frac{t^3}{3} + \frac{t^5}{5} \right) + C
\]

\[
= \frac{\tan^3(2x)}{6} + \frac{\tan^5(2x)}{10} + C
\]

where \(C\) is the constant of integration. The definite integral then has value

\[
\int_{\pi/8}^{\pi/2} \tan^2(2x) \sec^4(2x) \, dx = \left[ \frac{\tan^3(2x)}{6} + \frac{\tan^5(2x)}{10} \right]_{\pi/8}^{\pi/2}
\]

\[
= \left[ \frac{\tan^3(\pi)}{6} + \frac{\tan^5(\pi)}{10} \right] - \left[ \frac{\tan^3(\pi/4)}{6} + \frac{\tan^5(\pi/4)}{10} \right]
\]

\[
= -\frac{4}{15}.
\]
Alternatively, replace $\tan^2(2x) \text{ with } \sec^2(2x) - 1$ and use Formula 21.

(b) The indefinite integrals are integrated as follows:

(i) In this integral note that $d(x^3) = 3x^2dx$ to make the initial substitution $u = x^3$

$$\int \frac{x^2dx}{(16 - x^6)^{1/2}} = \frac{1}{3} \int \frac{du}{((4^2 - u^2)^{1/2}} \text{ set } u = x^3$$

$$= \frac{1}{3} \int \frac{4 \cos(\theta)d\theta}{((4^2 - (4^2 \sin^2(\theta)))^{1/2}} \text{ set } u = 4\sin(\theta)$$

$$= \frac{1}{3} \theta + C$$

$$= \frac{1}{3} \arcsin \left(\frac{x^3}{4}\right) + C,$$

where $C$ is a constant of integration.

Note that the integral $\int \frac{du}{((4^2 - u^2)^{1/2}}$ could have evaluated using Formula 28.

(ii) We must first "complete the square" for the given quadratic in the numerator. Thus

$$x^2 + 2x + 10 = (x + 1)^2 + (3^2).$$

This indicates that we should use the substitution $t = x + 1$ with $dx = dt$ to obtain

$$\int \frac{(x + 2)dx}{(x^2 + 2x + 10)} = \int \frac{(t + 1)dt}{((t^2 + (3)^2)}.$$

This integral can now be evaluated using Formula 25 with $b = 1, c = 1$ and $a = 3$. Alternatively, we make the substitutions $u = t^2 + 9$ with $du = 2tdt$ and $t = 3\tan(\theta)$ with $dt = 3\sec^2(\theta)d\theta$:

$$\int \frac{(x + 2)dx}{(x^2 + 2x + 10)} = \int \frac{(t + 1)dt}{((t^2 + (3)^2)}$$

$$= \frac{1}{2} \int \frac{(2t)dt}{((t^2 + (3)^2)} + \int \frac{dt}{((t^2 + (3)^2)}$$

$$= \frac{1}{2} \int \frac{du}{u} + \int \frac{3\sec^2(\theta)d\theta}{((9\tan^2(\theta) + 9)}$$

$$= \frac{1}{2} \ln|u| + \frac{1}{3} \int d\theta = \frac{1}{2} \ln(|u|) + \frac{1}{3} \theta + C$$

$$= \frac{1}{2} \ln |(x + 1)^2 + 9| + \frac{1}{3} \arctan \left(\frac{t}{3}\right) + C$$

$$= \frac{1}{2} \ln |(x + 1)^2 + 9| + \frac{1}{3} \arctan \left(\frac{x + 1}{3}\right) + C,$$

where $C$ is an arbitrary constant of integration.
QUESTION 2

(i) This indefinite integral is evaluated using the method of partial fractions. Since the integrand is a rational polynomial, by the method of partial fractions there exists constants $A$, $B$ and $C$ such that

\[
\frac{2x^2}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}.
\]

We use the method of undetermined coefficients to find the constants. Rewriting the right hand side of this equation on a common denominator we have

\[
\frac{2x^2}{(x^2 + 1)(x - 1)} = \frac{(Ax + B)(x - 1) + C(x^2 + 1)}{(x^2 + 1)(x - 1)} = \frac{(A + C)x^2 + (-A + B)x + (C - B)}{(x^2 + 1)(x - 1)}.
\]

Equating coefficients of the polynomial in the numerator of both sides, constants $A$, $B$ and $C$ must satisfy the three equations

\[
A + C = 2, \quad \text{and} \quad -A + B = 0, \quad \text{and} \quad -B + C = 0.
\]

Hence $A = 1$, $B = 1$ and $C = 1$, and

\[
\frac{2x^2}{(x^2 + 1)(x - 1)} = \frac{x + 1}{x^2 + 1} + \frac{1}{x - 1}.
\]

The integral can now be evaluated

\[
\int \frac{2x^2}{(x^2 + 1)(x - 1)} \, dx = \int \frac{x + 1}{x^2 + 1} \, dx + \int \frac{1}{x - 1} \, dx
\]

\[
= \int \frac{x}{x^2 + 1} \, dx + \int \frac{1}{x^2 + 1} \, dx + \int \frac{1}{x - 1} \, dx
\]

\[
= \frac{1}{2} \ln |x^2 + 1| + \tan^{-1}(x) + \ln |x - 1| + C,
\]

where $C$ is an arbitrary constant of integration. Check these integrations using the formula sheet.

(ii) The integral is integrated using the method of integration by parts or by directly applying Formula 9.

\[
\int e^{-4x} \cos(x) \, dx = \int e^{-4x} d(\sin(x))
\]

\[
= e^{-4x} \sin(x) - \int \sin(x) d(e^{-4x}) = e^{-4x} \sin(x) + 4 \int \sin(x) e^{-4x} \, dx
\]

\[
= e^{-4x} \sin(x) + 4 \int e^{-4x} d(-\cos(x))
\]

\[
= e^{-4x} \sin(x) + 4 \left[ -e^{-4x} \cos(x) + \int \cos(x) d(e^{-4x}) \right]
\]

\[
= e^{-4x} \sin(x) + 4 \left[ -e^{-4x} \cos(x) + 4 \int e^{-4x} \cos(x) \, dx \right].
\]

Hence collecting integral terms

\[
\int e^{-4x} \cos(x) \, dx = \frac{1}{17} \left[ e^{-4x} \sin(x) - 4e^{-4x} \cos(x) \right].
\]
(iii) If we apply the hint given that

\[ \cos(x) = 2 \cos^2 \left( \frac{x}{2} \right) - 1, \]

the integral can be simply evaluated as follows:

\[
\int \frac{dx}{1 + \cos(x)} = \int \frac{dx}{1 + 2 \cos^2 \left( \frac{x}{2} \right) - 1} = \int \frac{dx}{2 \cos^2 \left( \frac{x}{2} \right)}
\]

\[ = \frac{1}{2} \int \sec^2 \left( \frac{x}{2} \right) dx = \int \sec^2(u) du \quad \text{set } u = x/2 \text{ with } du = dx/2
\]

\[ = \tan(u) + C = \tan \left( \frac{x}{2} \right) + C,
\]

where \( C \) is a constant of integration.

**QUESTION 3**

(a) The temperature \( T \) at a distance \( r \) from the common centres of the spheres satisfies the second order differential equation

\[
\frac{d^2T}{dr^2} + \frac{2}{r} \frac{dT}{dr} = 0.
\]

If as requested we set \( S = \frac{dT}{dr} \), then this equation simply becomes a first order differential equation for \( S \) as a function of \( r \), namely

\[
\frac{dS}{dr} + \frac{2}{r} S = 0.
\]

This first order differential equation is of “integrating factor type” and has the integrating factor

\[ I = \exp \left( \int \frac{2}{r} dr \right) = \exp(2\ln|r|) = r^2,
\]

ignoring constants of integration. Multiplying both sides of the differential equation by the integrating factor \( I \) the left hand side converts to the derivative of the product \( SI \), namely

\[
r^2 \frac{dS}{dr} + (r^2) \frac{2}{r} S = 0,
\]

becomes

\[
\frac{d (r^2 S)}{dr} = 0.
\]

Integrating with respect to \( r \), we have that

\[ r^2 S(r) = C_1, \quad \text{or } S(r) = \frac{C_1}{r^2},
\]

where \( C_1 \) is a constant of integration.

Alternatively, this is a separable equation for \( S \) as a function of \( r \) - separating variables and integrating:

\[
\int \frac{dS}{S} = -2 \int \frac{dr}{r},
\]

\[ \ln |S| = -2 \ln |r| + C.
\]
Taking exponentials, \( S(r) = \frac{C_1}{r^2} \) for some constant \( C_1 \) (\( C_1 \) is derived from the constant of integration, \( C \)).

Now
\[
S(r) = \frac{dT}{dr} = \frac{C_1}{r^2}.
\]

Integrating this first order differential equation for \( T \), we find that
\[
T(r) = -\frac{C_1}{r} + C_2,
\]

where \( C_2 \) is another constant of integration. These two constants of integration are determined by the two boundary conditions
\[
T(1) = 15^\circ C \quad \text{and} \quad T(2) = 25^\circ C.
\]

Using these two conditions we find \( C_1 \) and \( C_2 \) satisfy the two equations
\[
15 = -C_1 + C_2 \quad \text{and} \quad 25 = -\frac{c_1}{2} + C_2.
\]

The solutions are easily found
\[
C_1 = 20 \quad \text{and} \quad C_2 = 35.
\]

Hence the temperature \( T \) at any \( r \) is given by the formula
\[
T(r) = -\frac{20}{r} + 35 \ (^\circ C).
\]

(b)

(i) The typical first order decay equation is of the form
\[
\frac{dm}{dt} = -km,
\]

where \( m \) is the mass of the material and \( k \) is the decay rate (positive constant). The solution to this separable equation is given by
\[
m = m_0e^{-kt},
\]

where \( m_0 \) is the initial mass.

It is only left to determine the value of the decay constant \( k \). We are given that the half life of radium is \( T_{1/2} = 1590 \) years - this provides us with a specific condition that enables us to solve for the unknown \( k \). Thus
\[
m(1590) = \frac{m_0}{2} = m_0e^{-1590k},
\]
\[
\frac{1}{2} = e^{-1590k},
\]
\[
-\ln(2) = -1590k,
\]
\[
k = \frac{\ln(2)}{1590},
\]
\[
\approx 0.00043594.
\]
(ii) We are required to find the time $t_f$ at which an initial mass $m_0 = 100\text{mg}$ decays to a weight of $m(t_f) = 30\text{mg}$. Using the above equation,

\[
30 = 100 \exp(-\ln(2) t_f/1590),
\]

\[
\ln(3/10) = (-\ln(2) t_f/1590),
\]

\[
t_f = \frac{-1590 \ln(3/10)}{-\ln(2)},
\]

\[
\approx 2761.8 \text{ years}.
\]

**QUESTION 4**

(a) Simpson’s Rule with $n = 6$ is of the form

\[
S_6 = \frac{\Delta x}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6].
\]

Now

\[
\Delta x = \frac{(2 - 1)}{6} = 1/6,
\]

\[
y_0 = \frac{1}{(1 + 1)} = 1/2,
\]

\[
y_1 = \frac{1}{(1 + 7/6)} = 6/13,
\]

\[
y_2 = \frac{1}{(1 + 8/6)} = 6/14,
\]

\[
y_3 = 6/15,
\]

\[
y_4 = 6/16,
\]

\[
y_5 = 6/17,
\]

\[
y_6 = 6/18.
\]

Therefore

\[
S_6 = \frac{1}{18} \left[ \frac{1}{2} + 4 \frac{6}{13} + 2 \frac{6}{14} + 4 \frac{6}{15} + 2 \frac{6}{16} + 4 \frac{6}{17} + \frac{6}{18} \right],
\]

\[
\approx 0.405466374.
\]

To estimate the error bound on Simpson’s rule, we are required to find an upper bound estimate of $|f^{(4)}(x)|$ on the interval of integration (from $1 \to 2$).

Given $f(x) = 1/(1 + x)$ we then have
\[ f^{(1)}(x) = -\frac{1}{(1 + x)^2}, \]
\[ f^{(2)}(x) = \frac{2}{(1 + x)^3}, \]
\[ f^{(3)}(x) = -\frac{6}{(1 + x)^4}, \]
\[ f^{(4)}(x) = \frac{24}{(1 + x)^5}, \]

Note that \( f^{(4)}(x) \) is positive and always decreasing on the interval of integration. Consequently, it will be at a maximum for \( x = 1 \) where \( f^{(4)}(1) = \frac{3}{4} \). Thus the error is bounded by

\[
\left| \int_1^2 \frac{1}{1 + x} \, dx - S_6 \right| \leq \frac{|f^{(4)}(x)(b - a)|}{180n^4},
\]
\[
\leq \frac{3(2 - 1)^5}{4(180)6^4},
\]
\[
\approx 0.000003.
\]

Consequently the approximation should be accurate to approximately 5 decimal places.

(b) The technique to solve this problem was not covered in the course material for the year.

**QUESTION 5**

(a) Let

\[ I = \int \cos^n(x) \, dx. \]

If we rewrite the integral in the form,

\[ I = \int \cos(x) \cos^{n-1}(x) \, dx, \]

we may then proceed to resolve the reduction formula through integration by parts.

Let

\[ u = \cos^{n-1}(x), \quad dv/dx = \cos(x), \]
\[ du/dx = -(n - 1) \sin(x) \cos^{n-2}(x), \quad v = \sin(x). \]
Then

\[ I = \sin(x) \cos^{n-1}(x) + (n - 1) \int \sin^2(x) \cos^{n-2}(x) dx, \]

\[ = \sin(x) \cos^{n-1}(x) + (n - 1) \int (1 - \cos^2(x)) \cos^{n-2}(x) dx, \]

\[ = \sin(x) \cos^{n-1}(x) + (n - 1) \int \cos^{n-2}(x) dx - (n - 1) I, \]

\[ nI = \sin(x) \cos^{n-1}(x) + (n - 1) \int \cos^{n-2}(x) dx, \]

\[ I = \frac{\sin(x) \cos^{n-1}(x)}{n} + \frac{(n - 1)}{n} \int \cos^{n-2}(x) dx. \]

(b) The volume of the solid of revolution is found by taking circular slices (refer to Figure 1) with width \( dx \) and area \( A(x) \) where

\[ A(x) = \pi r^2 = \pi (\cos^3 x)^2 = \pi \cos^6 x. \]

Thus the volume of the solid of revolution is

![Figure 1: Solid of Revolution formed by \( \cos^3 x \)](image)

\[ V = \int A(x) dx = \int_{-\pi/2}^{\pi/2} \pi \cos^6(x) dx. \]

Using the reduction formula,

\[ \int_{-\pi/2}^{\pi/2} \cos^6(x) dx = \frac{\sin(x) \cos^5(x)}{6} \bigg|_{-\pi/2}^{\pi/2} + \frac{5}{6} \int_{-\pi/2}^{\pi/2} \cos^4(x) dx, \]

\[ = \frac{5}{6} \int_{-\pi/2}^{\pi/2} \cos^4(x) dx, \]
and repeating the procedure noting that in this case the initial term in the reduction formula always drops to zero upon insertion of the limits ...

\[
\int_{-\pi/2}^{\pi/2} \cos^2(x) \, dx = \frac{5}{4} \left( \int_{-\pi/2}^{\pi/2} dx \right) - \frac{3}{4} \left( \int_{-\pi/2}^{\pi/2} \right) = \frac{5}{16} \pi.
\]

(c) We are given the parametric curves 

\[ x(t) = 3t - 6t^2, \quad \text{and} \quad y(t) = 8t^{3/2}, \]

valid over the interval \( t \in [1, 4] \).

To find the length of the arc, we proceed with the following formula,

\[
\left( \frac{ds}{dt} \right)^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2.
\]

Now

\[
\frac{dx}{dt} = 3 - 12t, \quad \text{and} \quad \frac{dy}{dt} = 12t^{1/2}.
\]

Thus

\[
\left( \frac{ds}{dt} \right)^2 = \left[ (3 - 12t)^2 + (12t^{1/2})^2 \right],
\]

\[= 9 - 72t + 144t^2 + 144t,
\]

\[= 9 + 72t + 144t^2,
\]

\[= (3 + 12t)^2,
\]

\[
\frac{ds}{dt} = |3 + 12t|, \quad \text{and} \quad (3 + 12t > 0 \text{ for } 1 \leq t \leq 4)
\]

Integrating,

\[ s = \int_{1}^{4} (3 + 12t) \, dt = [3t + 6t^2]_1^4 = 12 + 96 - 3 - 6 = 99. \]

The length of the arc is 99 units long.

**QUESTION 6**

(a) The Maclaurin polynomial of degree three is of the form

\[ M(x) = F(0) + F^{(1)}(0)x + \frac{F^{(2)}(0)}{2} x^2 + \frac{F^{(3)}(0)}{6} x^3. \]

Note that we never need to calculate \( F(x) \) explicitly for non-zero values of \( x \) and thus we never need to solve the integral itself (here \( F(0) = 0 \), the trivial solution to an integral for which the interval of integration is of zero length).
By the Fundamental Theorem of Calculus, the derivative of $F(x)$ is given by
\[ F^{(1)}(x) = \sin(x^2) + \cos(x), \]
\[ \Rightarrow \quad F^{(1)}(0) = 1. \]

Furthermore,
\[ F^{(2)}(x) = 2x \cos(x^2) - \sin(x), \]
\[ \Rightarrow \quad F^{(2)}(0) = 0. \]
\[ F^{(3)}(x) = 2 \cos(x^2) - 4x^2 \sin(x^2) - \cos(x), \]
\[ \Rightarrow \quad F^{(3)}(0) = 1. \]

Thus the Maclaurin series of degree three for $F(x)$ is
\[ M(x) = x + \frac{x^3}{6}. \]

Consequently, a suitable approximation for $F(0.5)$ is given by
\[ F(0.5) \approx M(0.5) = \frac{1}{2} + \frac{1}{48} = \frac{25}{48}. \]

(b) We are required to find the volume of the solid of revolution for the region bounded by $y = 1$, $x = 1$, and $y = e^x + 1$ about the line $x = 1$. Refer to Figure 2. Taking vertical strips of width $dx$ we shall rotate these strips around the $x = 1$ axis to form cylindrical shells. The total volume will be the summation (integration) of the volume of each of these shells.

Each shell has width $dx$ and surface area
\[ A(x) = [2\pi(1-x)] \ast (e^x + 1 - 1) = 2\pi e^x(1-x). \]
The volume of this shell is given by

\[ dV = A(x)dx, \]
\[ = 2\pi e^x(1 - x)dx. \]

Integrating,

\[ V = \int_{-\infty}^{1} 2\pi e^x(1 - x)dx, \]
\[ = 2\pi \lim_{t \to -\infty} \left[ \int_{t}^{1} e^x(1 - x)dx \right], \]
\[ = 2\pi \lim_{t \to -\infty} \left[ \int_{t}^{1} e^x dx - \int_{t}^{1} xe^x dx \right], \]
\[ = 2\pi \lim_{t \to -\infty} \left[ e^x - xe^x + e^x \right]_t, \]
\[ = 2\pi \lim_{t \to -\infty} \left[ e^x(2 - x) \right]_t, \]
\[ = 2\pi \lim_{t \to -\infty} \left[ e - (e^t(2 - t)) \right] = 2\pi e. \]

The volume of the solid of revolution is then \( V = 2\pi e \approx 17.079. \)
QUESTION 7

(a) Let \( a_k = \frac{k^2}{k^2 + 1} \).

Note that \( a_0 = 0 \), so we need only consider the series from \( k = 1 \). Now

\[
a_k = \frac{k^2}{k^2 + 1} = \frac{1}{1 + (1/k^2)} \geq \frac{1}{2},
\]

for all \( k \geq 1 \), thus

\[
\sum_{k=1}^{\infty} \frac{1}{2} \leq \sum_{k=1}^{\infty} a_k.
\]

Since the first series diverges, then by the comparison test, the second series must also diverge.

(b) Let \( T > 2 \) be some finite number and consider the proper integral

\[
\int_2^T \frac{1}{x(\ln(x))^2} \, dx.
\]

Making the substitution \( u = \ln(x) \), where \( du = (1/x)\,dx \), we have

\[
\int_2^T \frac{1}{x(\ln(x))^2} \, dx = \int_{u(2)}^{u(T)} \frac{du}{u^2} = \left. \frac{-1}{\ln(u)} \right|_2^T = \frac{1}{\ln(2)} - \frac{1}{\ln(T)}.
\]

Now by taking the limit as \( T \to \infty \) we may determine if the improper integral is convergent or divergent.

\[
\lim_{T \to \infty} \int_2^T \frac{1}{x(\ln(x))^2} \, dx = \lim_{T \to \infty} \left( \frac{1}{\ln(2)} - \frac{1}{\ln(T)} \right) = \frac{1}{\ln(2)}.
\]

Since the limit is finite, the integral is convergent.

(c) Elements within horizontal circular discs as illustrated in Figure 3 all require the same work to lift them to the required height above the reservoir. Consequently we may simplify the problem to summing the work required to lift each disc to the specified height of 1m above the reservoir.

The volume of water within a circular disc is represented by

\[
dV = \pi x^2 \, dy,
\]

\[
= \pi (9 - y^2) \, dy.
\]

The work required to move this disc \((-y + 1)\)m vertically is given by

\[
dW = \rho dV (-y + 1),
\]

\[
= \rho \pi (9 - y^2) (-y + 1) \, dy.
\]
where $\rho$ is the pressure density (here $\rho = 100 \text{N/m}^3$). Therefore the total work required is

\[
W = \int_{-3}^{0} dW, \\
= \int_{-3}^{0} \rho \pi (9 - y^2)(-y + 1) dy, \\
= \rho \pi \int_{-3}^{0} (y^3 - y^2 - 9y + 9) dy, \\
= \rho \pi \left[ \frac{y^4}{4} - \frac{y^3}{3} - \frac{9y^2}{2} + 9y \right]_{-3}^{0}, \\
= \rho \pi \left[ \frac{81}{4} - 9 + \frac{81}{2} + 27 \right], \\
= 3825\pi \text{ Joules.}
\]