1. Classify the following partial differential equations as parabolic, hyperbolic or elliptic

(i) \[ \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0 \]

(ii) \[ (c^2 - 1) \frac{\partial^2 u}{\partial x^2} - 2c^2 \frac{\partial^2 u}{\partial x \partial y} + (c^2 + 1) \frac{\partial^2 u}{\partial y^2} = 2 \frac{\partial u}{\partial x} \]

(iii) \[ u \frac{\partial^2 u}{\partial x^2} + 2c \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^2 u}{\partial y^2} = c^2 - u^2 \]

3. Consider the situation in which heat is being generated uniformly within a long thin insulated bar whose ends are maintained at constant temperature. Suitable scaling allows us to assume the bar to be of unit length and the scaled temperature \( u \) to be governed by the equations

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1 \]

\[ u(x, 0) = x \quad 0 \leq x \leq 1 \]
\[ u(0, t) = 0 \quad t \geq 0 \]
\[ u(1, t) = 1 \quad t \geq 0. \]

Show that the problem may be approximated, in the usual notation, by the explicit scheme

\[ u_{i,j+1} = u_{i,j} + \frac{k}{h^2} (u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + h^2). \]

Taking \( h = 0.2 \) and \( k = 0.01 \), and retaining four decimal places in your working, generate the solution at the first four time steps.

What is the steady state temperature of the bar?
1. (i) For the equation
\[
\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0
\]
we have \( b^2 - ac = 0 \) and so the problem is parabolic.

(ii) For
\[
(c^2 - 1) \frac{\partial^2 u}{\partial x^2} - 2c^2 \frac{\partial^2 u}{\partial x \partial y} + (c^2 + 1) \frac{\partial^2 u}{\partial y^2} = 2 \frac{\partial u}{\partial x}
\]
we have \( b^2 - ac = 1 \) and so the problem is hyperbolic.

(iii) For
\[
u \frac{\partial^2 u}{\partial x^2} + 2\epsilon \frac{\partial^2 u}{\partial x \partial y} + u \frac{\partial^2 u}{\partial y^2} = c^2 - u^2
\]
we have \( b^2 - ac = c^2 - u^2 \) and the problem is elliptic for \( u^2 > c^2 \), parabolic for \( u^2 = c^2 \) and hyperbolic for \( u^2 < c^2 \).

3. Noting that the formula for \( u_{i,j+1} \) is explicit, we approximate to the terms of the equation using
\[
\frac{\partial u}{\partial t} \approx \frac{u_{i,j+1} - u_{i,j}}{k}
\]
\[
\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}
\]
The formula for \( u_{i,j+1} \) is readily found after substitution of the above expressions and rearrangement.

Setting \( h = 0.2 \) and \( k = 0.01 \) we obtain the update equation
\[
u_{i,j+1} = 0.25u_{i+1,j} + 0.5u_{i,j} + 0.25u_{i-1,j} + 0.01.
\]
The solution at the first four time rows is set out below.

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<th>t</th>
<th>x = 0.0</th>
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<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
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</tbody>
</table>

The steady state temperature is found by setting \( \frac{\partial u}{\partial t} = 0 \) giving \( u_{xx} + 1 = 0 \). Solving, and allowing for the end conditions, we obtain \( u = \frac{1}{2}(3t - t^2) \).