Test 2 Solutions

1. Using first the identity \( \sin(A + B) = \cos A \sin B + \sin A \cos B \) with \( A = 2\theta \) and \( B = \pi/2 \) gives

\[
\sin(2\theta + \pi/2) = \cos 2\theta \sin \pi/2 + \sin 2\theta \cos \pi/2 = \cos 2\theta
\]

(where we have used \( \cos \pi/2 = 0 \) and \( \sin \pi/2 = 1 \)). Next the identity \( \cos 2A = \cos^2 A - \sin^2 A \) with \( A = \theta \) gives

\[
\sin(2\theta + \pi/2) = \cos 2\theta = \cos^2 \theta - \sin^2 \theta.
\]

2. Substituting \( \cos^2 x = 1 - \sin^2 x \) we have

\[
2 \cos^2 x - \sin x = 2(1 - \sin^2 x) - \sin x = -2\sin^2 x - \sin x + 2 = 1
\]

or \( -2\sin^2 x - \sin x + 1 = (\sin x + 1)(-2 \sin x + 1) = 0 \). Thus \( \sin x = -1 \) or \( \sin x = \frac{1}{2} \). For \( 0 \leq x \leq 360^\circ \) \( \sin x = -1 \) gives \( x = 270^\circ \) while \( \sin x = \frac{1}{2} \) gives \( x = 30^\circ \) and \( x = 150^\circ \). The required solutions are thus \( x = 30^\circ \), \( x = 150^\circ \) and \( x = 270^\circ \).

3. We write

\[
4 \cos 2t + 3 \sin 2t = A \sin(2t - \phi)
\]

\[
= A(\sin 2t \cos \phi - \cos 2t \sin \phi)
\]

\[
= A \sin 2t \cos \phi - A \cos 2t \sin \phi.
\]

Comparing coefficients of \( \cos 2t \) we have \( A \sin \phi = -4 \) and of \( \sin 2t \) we have \( A \cos \phi = 3 \). Squaring and adding gives \( A^2 = 25 \) and so we take \( A = 5 \). Dividing these two equations we have \( \tan \phi = -\frac{4}{3} \) or \( \phi = -\tan^{-1}\frac{4}{3} = -0.9273 \). We note that \( \sin \phi \) is negative while \( \cos \phi \) is positive so that \( \phi \) lies in the fourth quadrant so that the value \( \phi = -0.9273 \) is correct. The required expression is thus \( 5 \sin(2t + 0.9273) \).

4. We here have simply \( \sin A = \frac{7}{12} \) so that \( A = \sin^{-1} \frac{7}{12} = 35.68^\circ \) or 0.6228 radians.

5. We first note that \( B = 180 - A - C = 180 - 45 - 55 = 80^\circ \). Next the sine rule gives
\[
b = \frac{\sin B}{\sin A} = \frac{4 \sin 80^\circ}{\sin 45^\circ} = 5.571
\]

and

\[
c = \frac{\sin C}{\sin A} = \frac{4 \sin 55^\circ}{\sin 45^\circ} = 4.634.
\]

6. On the first bearing the yacht travels \(30 \sin 60^\circ = 25.98\) nautical miles east and \(30 \cos 60^\circ = 15\) nautical miles north. On the second bearing the yacht travels \(48 \sin 60^\circ = 41.57\) nautical miles east and \(48 \cos 60^\circ = 24\) nautical miles south. The combined distance travelled is then \(25.98 + 41.57 = 67.55\) miles east and \(24 - 15 = 9\) miles south from the starting point.

We next calculate the bearing as \(-\tan^{-1} \frac{07.55}{9} = -82.41^\circ\) and the distance back to the starting point is 68.15 miles. Thus it will take 9.735 hours or 9 hours 44 minutes to return to the starting point at seven knots.

7. (i) \(3\mathbf{a} + \mathbf{b} = 3(2, -1, 1) + (1, 2, 3) = (6, -3, 3) + (1, 2, 3) = (7, -1, 6)\).

(ii) Now \(\mathbf{a} - \mathbf{b} = (1, -3, -2)\) so that \(|\mathbf{a} - \mathbf{b}| = \sqrt{1 + 9 + 4} = \sqrt{14}\).

(iii) First \(|\mathbf{b}| = \sqrt{14}\) so that \(\hat{\mathbf{b}} = \frac{1}{\sqrt{14}}(1, 2, 3)\).

8. We have for the dot product \(\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta\) where \(\theta\) is the angle between \(\mathbf{a}\) and \(\mathbf{b}\). We therefore have

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-6}{5\sqrt{5}} = -0.5367.
\]

This gives \(\theta = -57.54^\circ\) or \(-1.0043\) radians. More appropriately we have \(\theta = 122.46^\circ\) or 2.1373 radians.
9. In the usual way we calculate $\mathbf{g} \times \mathbf{h}$ from the array

\[
\begin{array}{ccc}
i & j & k \\
2 & 1 & 0 \\
-1 & 0 & 3
\end{array}
\begin{array}{ccc}
i & j & k \\
2 & 1 & 0 \\
-1 & 0 & 3
\end{array}
\]

and obtain $\mathbf{g} \times \mathbf{h} = 3\mathbf{i} - 6\mathbf{j} + \mathbf{k}$.

To show that $\mathbf{g} \times \mathbf{h}$ is perpendicular to both $\mathbf{g}$ and $\mathbf{h}$ we merely have to show that $(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{g}$ and $(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{h}$ are each zero. This is easily done:

\[
(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{g} = (3, -6, 1) \cdot (2, 1, 0) = 0
\]

\[
(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{h} = (3, -6, 1) \cdot (-1, 0, 3) = 0.
\]

10. The equation of the plane passing through the point $A$ with position vector $\mathbf{a}$ and having normal $\mathbf{n}$ is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ or $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$. With $\mathbf{a} = (1, 2, -1)$ and $\mathbf{n} = (2, 3, -1)$ we have

\[
\mathbf{r} \cdot (2, 3, -1) = (1, 2, -1) \cdot (2, 3, -1) = 9
\]

or with $\mathbf{r} = (x, y, z)$ we have the equation in cartesian form as $2x + 3y - z = 9$.

Considering the point with position vector $(2, 1, 0)$, that is the point $x = 2$, $y = 1$, $z = 0$, we have $2x + 3y - z = 7 \neq 3$ so that the point does not lie on the plane.