Test 2 Solutions

1. The identity \( \cos (A - B) = \cos A \cos B + \sin A \sin B \) with \( A = 90^\circ \) and \( B = \theta \) gives

\[
\cos(90^\circ - \theta) = \cos 90^\circ \cos \theta + \sin 90^\circ \sin \theta
\]
\[
= 0 \times \cos \theta + 1 \times \sin \theta
\]
\[
= \sin \theta.
\]

2. We first note that \( 0 \leq x \leq 2\pi \) implies \( 0 \leq 2x \leq 4\pi \). Now using a calculator \( \cos 2x = 0.4 \) implies \( 2x = 1.1593 \) radians while the other value for \( 2x \) in the first revolution is \( 2\pi - 1.1593 = 5.1239 \). The two values in the second revolution (again for \( 2x \)) are obtained by adding \( 2\pi \) to these two values. The four required values for \( x \) are then 0.5796, 2.5620, 3.7212 and 5.7035.

3. We write

\[
2 \cos 5t - 3 \sin 5t = A \sin(5t - \phi)
\]
\[
= A(\sin 5t \cos \phi - \cos 5t \sin \phi)
\]
\[
= A \sin 5t \cos \phi - A \cos 5t \sin \phi.
\]

Comparing coefficients of \( \cos 5t \) and \( \sin 5t \) we have \( -A \sin \phi = 2 \) and \( -3 = A \cos \phi \). Squaring and adding we get \( A^2 = 13 \) and therefore take \( A = \sqrt{13} = 3.606 \). Dividing we obtain \( \tan \phi = \frac{2}{3} \) or \( \phi = \tan^{-1} \frac{2}{3} = 0.5880 \). Noting however that both \( \sin \phi \) and \( \cos \phi \) are negative we clearly require \( \phi \) to lie in the third quadrant and so take \( \phi = 0.5880 + \pi = 3.7296 \). The required expression is thus \( \sqrt{13} \sin(5t - 3.73) \).

4. Referring to figure 1 we have \( \sin A = \frac{6}{11} \) so that \( A = \sin^{-1} \frac{6}{11} = 0.5769 \) radians (or 33.06° degrees).

5. Referring to figure 2 we first have \( B = 10 - A - C = 80^\circ \). Next the sine rule gives

\[
b = a \frac{\sin B}{\sin A} = 6 \frac{\sin 80^\circ}{\sin 53^\circ} = 7.399
\]

and

\[
c = a \frac{\sin C}{\sin A} = 6 \frac{\sin 47^\circ}{\sin 53^\circ} = 5.495.
\]
6. Refer to figure 3. On the first bearing the yacht travels \(32 \sin 45^\circ = 22.627\) nautical miles east and \(32 \cos 45^\circ = 22.627\) nautical miles north. On the second bearing the yacht travels \(36 \sin 35^\circ = 20.649\) nautical miles west and \(36 \cos 35^\circ = 29.489\) nautical miles north. The combined distance travelled is then \(22.627 - 20.649 = 1.978\) miles east and \(22.627 + 29.489 = 52.12\) miles north from the starting point.

From the diagram we calculate the angle \(\phi\) to be \(\tan^{-1} \frac{1.978}{52.12} = 2.17^\circ\). The bearing required to return to the starting point is then \(-(180 - \phi) = -(180 - 2.17) = -177.83^\circ\). The distance \(AC\) is \(1.978/\sin 2.17 = 52.24\) miles which will take 7.46 hours or 7 hours 28 minutes at seven knots.

An alternative solution uses the cosine rule to give

\[
\begin{align*}
b &= \sqrt{a^2 + c^2 - 2ac \cos B} \\
&= \sqrt{(32^2 + 36^2 - 2 \times 32 \times 36 \cos 100^\circ)} \\
&= 52.17
\end{align*}
\]
and sine rule to give

\[ \sin A = \frac{36}{52.17} \sin 100 = 0.6796 \]

so that \( A = 42.81 \) and \( \phi = 45 - 42.81 = 2.19^\circ \) from which the bearing and duration of the return trip follow as before.

7. (i) \( \mathbf{a} + 3\mathbf{b} = (0, 4, 3) + 3(-1, 0, -1) = (0, 4, 3) + (-3, 0, -3) = (-3, 4, 0) \).
(ii) Now \( \mathbf{a} - \mathbf{b} = (1, 4, 4) \) so that \( | \mathbf{a} - \mathbf{b} | = \sqrt{1 + 16 + 16} = \sqrt{33} \).
(iii) First \( | \mathbf{a} | = 5 \) so that \( \hat{\mathbf{a}} = \frac{1}{5}(0, 4, 3) = (0, \frac{4}{5}, \frac{3}{5}) \).

8. We have for the dot product \( \mathbf{a} \cdot \mathbf{b} = | \mathbf{a} || \mathbf{b} | \cos \theta \) where \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \). We therefore have

\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{| \mathbf{a} || \mathbf{b} |} = \frac{5}{\sqrt{14} \sqrt{5}} = 0.5976. \]

so that \( \theta = 53.30^\circ \) or 0.9303 radians.
9. In the usual way we calculate \( \mathbf{g} \times \mathbf{h} \) from the array
\[
\begin{array}{ccc|ccc}
  i & j & k & i & j & k \\
  3 & 0 & -1 & 3 & 0 & -1 \\
  1 & 0 & 1 & 1 & 0 & 1 \\
\end{array}
\]
and obtain \( \mathbf{g} \times \mathbf{h} = -4\mathbf{j} \).

To show that \( \mathbf{g} \times \mathbf{h} \) is perpendicular to both \( \mathbf{g} \) and \( \mathbf{h} \) we merely have to show that \( (\mathbf{g} \times \mathbf{h}) \cdot \mathbf{g} \) and \( (\mathbf{g} \times \mathbf{h}) \cdot \mathbf{h} \) are each zero. This is easily done:
\[
(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{g} = (0, -4, 0) \cdot (3, 0, -1) = 0
\]
\[
(\mathbf{g} \times \mathbf{h}) \cdot \mathbf{h} = (0, -4, 0) \cdot (1, 0, 1) = 0.
\]

10. The equation of the plane passing through the point \( A \) with position vector \( \mathbf{a} \) and having normal \( \mathbf{n} \) is \( (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \) or \( \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \). With \( \mathbf{a} = (2, -1, 3) \) and \( \mathbf{n} = (-1, 1, 2) \) we have in vector form
\[
\mathbf{r} \cdot (-1, 1, 2) = (2, -1, 3) \cdot (-1, 1, 2) = 3
\]
ie \( \mathbf{r} \cdot (-1, 1, 3) = 3 \), or in cartesian form
\[
(x, y, z) \cdot (-1, 1, 3) = 3
\]
or \(-x + y + 2z = 3\).

It is readily seen that the point with position vector \((1, 0, 2)\), that is the point \( x = 1, y = 0, z = 2 \), satisfies the equation \(-x + y + 2z = 3\) and so does lie on the plane.