Test 1 Solutions

1. We have
   (i) \( f(g(x)) = f(2 - x^2) = 2(2 - x^2) + 1 = 5 - 2x^2, \)
   (ii) \( g(f(x)) = g(2x + 1) = 2 - (2x + 1)^2 = 1 - 4x - 4x^2. \)

2. A graph of the function \( f(x) = x^2 + 2x \) for \( 0 \leq x \leq 3 \) is shown in figure 1. It is clearly apparent from the graph that \( f \) is one-to-one. To find the inverse we have
   \[
   y = x^2 + 2x = (x + 1)^2 - 1
   \]
   so that \((x + 1)^2 = 1 + y\) or \(x + 1 = \sqrt{1 + y}\) or \(x = -1 + \sqrt{1 + y}\). Note that we have taken the positive square root as we require \(x \geq 0\). The inverse function is therefore \( y = -1 + \sqrt{1 + x} \) and is shown plotted in figure 2. The domain the function \( f \) is \([0, 3]\) and the range is \([0, 15]\). For the inverse function \( f^{-1}\) the domain is \([0, 15]\) and the range \([0, 3]\).

3. From \( x = t - 1 \) we have \( t = x + 1 \) so that \( y = 2t^2 + 1 = 2(x + 1)^2 + 1 = 2x^2 + 4x + 3. \) Also for \( 3 \leq t \leq 5 \) we have \( 2 \leq x \leq 4 \) and \( 19 \leq y \leq 51. \) It is clear that \( y \) is defined as a function of \( x \) and the domain of the function is \([2, 4]\) and the range is \([19, 51]\).

4. The function is shown in figure 3.
5. Given the solution $x = -1$ we have $x + 1$ as a factor of the cubic and $x^3 - 2x^2 - x + 2 = (x + 1)(ax^2 + bx + c) = 0$. Expanding brackets we have

$$x^3 - 2x^2 - x + 2 = (x + 1)(ax^2 + bx + c) = ax^3 + (a + b)x^2 + (b + c)x + c = 0$$

and comparing coefficients $a = 1$, $a + b = -2$, $b + c = -1$ and $c = 2$. Solving we have $a = 1$, $b = -3$ and $c = 2$ so

$$x^3 - 2x^2 - x + 2 = (x + 1)(x^2 - 3x + 2) == (x + 1)(x - 2)(x - 1)0$$

and the solutions are $x = -1$, $x = 1$ and $x = 2$.

6. Multiplying the first equations by two we have

$$2x - 4y = 10 \quad 2x + 3y = -4$$

and subtraction gives $7y = -14$ or $y = -2$. Substitution into either of the original equations now gives $x = 1$.

7. We write

$$\frac{2x + 1}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$= \frac{A(x + 2) + B}{(x + 2)^2} = \frac{Ax + (2A + B)}{(x + 2)^2}$$
Comparing coefficients we have $A = 2$ and $2A + B = 1$. These equations give $A = 2$ and $B = -3$ so that the final result is

$$\frac{2x + 1}{(x + 2)^2} = \frac{2}{x + 2} - \frac{3}{(x + 2)^2}.$$ 

8. Replacing the hyperbolic functions by their definitions we have

$$f(x) = \sinh^2 x + \cosh 2x = \left( e^x - e^{-x} \right)^2 + \frac{e^{2x} + e^{-2x}}{2}$$

$$= \frac{e^{2x} - 2 + e^{-2x}}{4} + \frac{e^{2x} - e^{-2x}}{2} = \frac{3}{4}(e^{2x} + e^{-2x}) - \frac{1}{2}$$

9. We have
   (i) $\log_2 8 = \log_2 2^3 = 3$,
   (ii) $\log_3 \frac{1}{5} = \log_3 3^{-2} = -2$
and
   (iii) $\log_2 3 \log_3 2 = \log_2 \frac{\log_2 2}{\log_2 3} = \log_2 2 = 1$.

10. Here $\log_4 2x = 3$ gives $2x = 4^3 = 64$ so that $x = 32$. 

Figure 3: Graph of $f(x)$ for question 3.