Trigonometric ratios

There are two standard ways of measuring angles. The most common is that of degrees which uses as reference the right angle which is taken to be 90°. With reference to figure 1, the angle between lines $AB$ and $AC$ is considered in relation to the right angle. Thus, for example, if the line $AC$ sweeps through half a right angle in moving from $AB$ to $AC$ then the angle $\theta$ is 45°, that is a half of 90°. Clearly if $AC$ sweeps out a complete revolution then this corresponds to 360°.

![Figure 1: Angle $\theta$ formed by lines $AB$ and $AC$.](image)

The second method for measuring angles, and that used most commonly in mathematics, is in terms of radians. In this case it is usual to consider the angle at the centre of a circle of radius $r$. Figure 2 shows an angle of one radian, the angle being such that the length of the arc subtended by the angle on the circumference of the circle is equal to the radius $r$. In this case, if an angle $\theta$ subtends an arc of length $R$ on the circumference of the circle radius $r$ then the value of $\theta$ in radians is $R/r$. In this case a complete revolution corresponds to $2\pi$.

Note that if we write $\theta = 35^\circ$ then the angle is measured in degrees while writing $\theta = 2.3$ the angle is measured in radians. Also to convert an angle $\theta$ measured in degrees to radians we have $\theta \pi / 180$.

Our next consideration is to define the trigonometric ratios. Consider the right triangle in figure 3 where the angle $C$ is a right angle. Relative to the
angle $\theta$ at A we refer to the sides as the hypotenuse, opposite and adjacent as shown in figure 3.

Figure 3: A right triangle $ABC$.

The trigonometric ratios of the angle $\theta$ are defined

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$$

These are respectively the sine, cosine and tangent. Whether the angle $\theta$ is measured in degrees or radians does not particularly concern us at the
moment (although strictly we should define different functions for each case). We note that these definitions, as derived by reference to the right triangle $ABC$, are valid for $0^\circ < \theta < 90^\circ$ (or $0 < \theta < \pi/2$).

It should be apparent from the triangle of figure 3 that (assuming angles are measured in degrees)

$$\sin(90 - \theta) = \cos \theta$$

$$\cos(90 - \theta) = \sin \theta$$

$$\tan(90 - \theta) = \left(\tan \theta\right)^{-1}.$$  

We also note that the tangent is simply

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

It is convenient to define the three additional trigonometric ratios

$$\sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta} \quad \cot \theta = \frac{1}{\tan \theta}$$

These are respectively the secant, cosecant and cotangent. We note that csc is sometimes written cosec.

Next we define the inverse trigonometric ratios. It should be apparent that each of the ratios considered can be inverted. That is, given the value of the ratio, the appropriate angle can be determined from a right triangle. Note that the size of the triangle is not a concern here as similar triangles will always have the same ratios. We thus define the inverse sine ratio as $\sin^{-1}$ such that given a value $x$, $\sin^{-1} x$ gives the angle $\theta$ whose sine is $x$. The inverse ratios from cos and tan are defined similarly. Note that in all cases we need decide beforehand whether the required angle is to be measured in degrees or radians.

Although we have here used $\sin^{-1}$ to denote the inverse of the sine ratio, you should be aware that alternative notations exist. In particular arcsin and asin are sometimes used in place of $\sin^{-1}$. Similarly $\cos^{-1}$ is sometimes written arccos or acos, and $\tan^{-1}$ is sometimes written arctan or atan. As an aside we also note that the inverse of the csc ratio might be written as
arccosec! The text however does not consider the inverses of the sec, csc and cot ratios.

The trigonometric ratios in all quadrants

The above discussion of the trigonometric ratios has been in the context of the right triangle so that the angle has been restricted to the interval $0^\circ < \theta < 90^\circ$ in degrees or $0 < \theta < \pi/2$ in radians. The intention now is to extend the definitions so that they are applicable to all values of $\theta$. We begin by noting that it is usual in this context to consider the plane to be made up of four quadrants as shown in figure 4.

![Figure 4: The four quadrants.](image)

We talk about an angle being in one of the four quadrants by drawing the angle from the horizontal line in an anti clockwise direction as indicated in figure 5. Thus if $\theta$ lies between $0^\circ$ and $90^\circ$ $\theta$ lies in the first quadrant, if $\theta$ lies between $90^\circ$ and $180^\circ$ $\theta$ lies in the second quadrant etc. In this way $\theta$ can take any positive value by allowing the angle to rotate about the origin in an anti clockwise direction, and any negative value by allowing the angle to rotate about the origin in a clockwise direction. Thus $\theta = 80^\circ$, $\theta = 440^\circ$ and $\theta = -280^\circ$ are essentially the same angle, and all lie in the first quadrant.

This arrangement allows us to use the previous definitions of the various ratios in terms of a right triangle by interpreting the hypotenuse, adjacent and opposite appropriately. Thus the adjacent is positive if measured to the right of the origin and negative if measured to the left of the origin, while the opposite is positive if measured above the origin and negative if measured below the origin. This is equivalent to taking adjacent and opposite to be the
usual $x$ and $y$ coordinates of a point lying on the line defining the angle. The hypotenuse is always positive. With this convention the ratios are defined for all values $\theta$. We note that it is often convenient here to imagine the angle being defined by a point on a circle centre the origin, so that as $\theta$ increases the point moves around the circumference of the circle.

With the above convention for determining the signs of the adjacent and opposite it is possible for the ratios to be positive or negative. The usual rule for recalling in which quadrant the sin, cos and tan ratios are positive is the ‘CAST’ rule.

With the convention that the adjacent is the $x$-component, the opposite is the $y$-component and the hypotenuse is the ‘radius’ $r$ the trigonometric ratios, for all $\theta$, become

\[
\sin \theta = \frac{y}{r} \]
\[
\cos \theta = \frac{x}{r} \]
\[
\tan \theta = \frac{y}{x}
\]

Consideration of the definition of sin for example in relation to the above discussion gives for example

\[
\sin(180^\circ - \theta) = \sin \theta.
\]
A number of other similar identities exist. As mentioned previously allowing an angle to complete a complete revolution of the circle leaves the angle essentially unchanged so that, for example,

\[ \cos(\theta + 360^\circ) = \cos \theta. \]

All trigonometric ratios are therefore periodic with period 360°.

Figures 6, 7 and 8 show plots of the ratios \( \sin \theta \), \( \cos \theta \) and \( \tan \theta \). The first two are plotted over three complete periods (revolutions). Note that the argument in each case is now assumed to be in radians. Also that \( \tan \theta \) has period \( \pi \).

Figure 6: The trigonometric ratio \( \sin \theta \).

Figure 7: The trigonometric ratio \( \cos \theta \).
Trigonometric identities

We conclude this section by considering a number of trigonometric identities. First

\[ \cos^2 x + \sin^2 x = 1 \]

\[ 1 + \tan^2 x = \sec^2 x \]

\[ 1 + \cot^2 x = \csc^2 x \]

may be obtained directly from Pythagoras’ theorem.

We next note the compound angle formulae

\[ \sin(x + y) = \sin x \cos y + \cos x \sin y \]

\[ \sin(x - y) = \sin x \cos y - \cos x \sin y \]

\[ \cos(x + y) = \cos x \cos y - \sin x \sin y \]

\[ \cos(x - y) = \cos x \cos y + \sin x \sin y \]

\[ \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \]
\[
\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}
\]

and the sum and product identities

\[
\sin x + \sin y = 2 \sin \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y)
\]

\[
\sin x - \sin y = 2 \sin \frac{1}{2} (x - y) \cos \frac{1}{2} (x + y)
\]

\[
\cos x + \cos y = 2 \cos \frac{1}{2} (x + y) \cos \frac{1}{2} (x - y)
\]

\[
\cos x - \cos y = -2 \sin \frac{1}{2} (x + y) \sin \frac{1}{2} (x - y).
\]

Note that most of these formulae are in fact repetitions using the fact that \(\sin(-x) = -\sin x, \cos(-x) = \cos x\) etc. For example, the following double-angle identities are readily derived from the above:

\[
\sin 2x = 2 \sin x \cos x
\]

\[
\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x.
\]