The exponential function

We are already familiar with the exponent notation \( b^a \) where \( b \) is the base and \( a \) is the exponent. Recall the basic rules for manipulating exponents:

\[
\begin{align*}
    b^m b^n &= b^{m+n} \\
    \frac{b^m}{b^n} &= b^{m-n} \\
    (b^m)^n &= b^{mn}.
\end{align*}
\]

A function of the form \( y = b^{ax} \), for given constant value \( a \) and \( b \) is an exponential function. Since all exponential functions \( y = b^{ax} \) are identical except for scaling and/or inversion of the \( x \) axis (when \( a < 0 \)) it is convenient to relate all to a standard exponential function, that is to the exponential function \( y = e^{ax} \), sometimes denoted \( y = \exp(ax) \), where \( e = 2.718\ldots \). The reason for using \( e^x \) will be clearer when you come to study the calculus. The exponential functions \( e^x \) and \( e^{-x} \) are plotted in figure 1. The exponential function obeys the usual rules for indices.

![Exponential functions](image)

Figure 1: Exponential functions \( e^x \) and \( e^{-x} \).
The functions $e^x$ and $e^{x/2}$ are plotted in figure 2.

![Figure 2: Exponential functions $e^x$ and $e^{x/2}$.](image)

A number of physical laws are such that some quantity decays exponentially over time, that is the quantity behaves like an exponential function $e^{at}$ for some $a$. When $a < 0$ the function decreases and we have exponential decay and when $a > 0$ the function increases and we have exponential growth. Examples of exponential decay is the variation with time of the radioactive material present in an isotope, or the amount of charge in a capacitor discharging through a resistor. An example of exponential growth would be the given in the situation in which a population of bacteria is growing without any constraint due to a lack of physical space or the availability of food.

A further example of exponential decay is provided by Newton’s law of cooling which states that the rate of loss of heat from a body is proportional to the temperature difference between the body and its surroundings. This leads to the equation

$$
\theta(t) = \theta_1 + (\theta_0 - \theta_1)e^{-kt}
$$

where $\theta(t)$ is the temperature of the body at time $t$, $\theta_0$ is the initial temperature of the body (ie at $t = 0$) and $\theta_0$ is the temperature of the surroundings.
The hyperbolic functions

It turns out that there is a close relationship between the trigonometric functions sin $x$ and cos $x$ (which we will study later, and sometimes called circular functions) and certain functions derived from the exponential function. To appreciate this relationship we define the following so called *hyperbolic functions.*

\[
\sinh x = \frac{e^x - e^{-x}}{2}
\]

\[
\cosh x = \frac{e^x + e^{-x}}{2}
\]

\[
\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]

These functions are the *hyperbolic sine,* *hyperbolic cosine* and *hyperbolic tangent* functions respectively and are plotted in figures 3, 4 and 5.

Figure 3: Hyperbolic sine function.
In addition we have the hyperbolic functions *cosech*, *sech* and *coth* defined

\[
\text{cosech } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}
\]

\[
\text{sech } x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}
\]

\[
\text{coth } x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}
\]

Note that there is some variation between texts in the notation used for cosech $x$ and sech $x$ in particular, although you do not need concern yourselves unduly about these functions. Care though needs be exercised when using a calculator to calculate values of the functions.
Among some of the identities that the hyperbolic functions satisfy are the following (see the text for more):

\[ \cosh^2 x - \sinh^2 x = 1 \]

\[ \cosh 2x = \cosh^2 x + \sinh^2 x \]

\[ 2 \sinh^2 x = \cosh 2x - 1 \]

The identities are usually best established by reference to their definition. The best strategy is usually to start with the more involved side of the identity and attempt to simplify it and rearrange to obtain the simpler side.

**Example** Consider \( 2 \sinh^2 x = \cosh 2x - 1 \).

Applying the definition of \( \sinh x \) to the left hand side we have

\[
2 \sinh^2 x = 2 \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

\[
= \frac{(e^x - e^{-x})^2}{2}
\]

\[
= \frac{e^{2x} - 2 + e^{-2x}}{2}
\]

\[
= \frac{e^{2x} + e^{-2x}}{2} - 1
\]

\[
= \cosh 2x - 1
\]

as required.