Parametric curves

The representation \( y = f(x) \) of a function is the cartesian form. An alternative representation in terms of an intermediate variable is also sometimes useful. Consider the situation in which we have \( x = f(t) \) and \( y = g(t) \) where \( f \) and \( g \) are given functions, and \( t \) is restricted to an interval \([a, b]\) say (where \( a \) and \( b \) are given constants). For any value \( t \) we have the related values \( x = f(t), y = g(t) \) which determines a point in the \( x-y \) plane. As \( t \) varies the point varies and traces out a curve in the \( x-y \) plane.

**Example** Consider the case \( x = 3t + 1, y = t^2, \ 0 \leq t \leq 1. \) We may here eliminate the variable \( t \) by rewriting \( x = 3t + 1 \) as \( t = \frac{1}{3}(x - 1) \) and obtain \( y = t^2 = \left(\frac{1}{3}(x - 1)\right)^2 = \frac{1}{9}(x - 1)^2 . \) Also the interval \([0, 1]\) for \( t \) corresponds to the interval \([1, 4]\) for \( x . \)

The equations \( x = f(t), y = g(t), \ a \leq t \leq b \) are referred to as parametric equations while \( t \) is the parameter. We note that when the equations define a function (as in the last example) the function is said to be in parametric form (the fact that the equations do not always represent a function is demonstrated in the next example).

**Example** Consider the equations \( x = \sin \theta, y = \cos \theta, \ 0 \leq \theta \leq 2\pi . \) It is a simple matter here, using the basic trigonometric identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to show that \( x^2 + y^2 = 1 \) so that points \((x, y)\) lie on a circle centre the origin, radius equal to one. In fact the curve traces out the circle once from the point \((1, 0)\) in a counter clockwise direction. It is clear that the equations do not represent a function.

It is not always possible as in this last example to eliminate the parameter to obtain an explicit expression for the function and even when possible, it is not always convenient. In such cases the use of a graphics calculator or computer software such as MATLAB will be found to be very useful.

**Example** Consider the equations \( x = t^2 - 1, y = (t + 2)^3 + 2, \ -2 \leq t \leq 2 . \) It is again possible here to eliminate \( t \) as we can write \( t = \sqrt{(x + 1)} . \) The expression \( y = (\sqrt{(x + 1)} + 2)^3 + 2 \) for \( y \) in terms of \( x \) is rather involved however. Using MATLAB we are able to obtain the curve shown below. We note that the curve is generated using the code

\[
t=-2:.01:2; \ x=t.^2-1; \ y=(t+2).^3+2; \ \text{plot}(x,y)
\]
Continuous and discontinuous functions

In general terms we say that a curve is continuous if it can be traced without lifting the pen from the paper. To be able to apply the term continuous mathematically to functions we need be rather more precise. We begin by defining the idea of a limit.

The limit of a function as $x$ tends a value $a$ say is quite obvious: it is the value of the function as $x$ approaches $a$, that is gets closer and closer to $a$. We need in fact be more specific and state whether $x$ is approaching $a$ from above or below and so write

$$\lim_{x \to a^-} f(x) \quad \lim_{x \to a^+} f(x)$$

respectively. The first limit is the limit as $x$ tends to $a$ from below (or from the left) the second is the limit as $x$ tends to $a$ from above (or from the right). To appreciate the finer points of limit we need consider a discontinuous function.

Example Consider the function

$$f(x) = \begin{cases} 
  x + 1 & x < 2 \\
  x^2 & x \geq 2 
\end{cases}$$

The graph is shown plotted below. Note the convention used regarding solid circles and open circles on the graph.
A function $f(x)$ is *continuous at a point* $x = a$ if the following holds

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(A).$$

**Periodic functions.**

A function is said to be *periodic* if $f(x + T) = f(x)$ for some value $T$, that is the value of the function repeats as the argument changes by a value $T$. The *period* of the function is the smallest such $T$ value. An example of a periodic function is given in the figure below.

Figure 2: Function with discontinuity at $x = 2$.

Figure 3: A periodic function.
**Even and odd functions.**

An *even function* is one such that \( f(-x) = f(x) \), that is changing the sign of the argument leaves the value of the function unchanged. From the figure below it is apparent that an even function is symmetric about the \( y \) axis.

![Figure 4: An even function.](image)

An *odd function* is one such that \( f(-x) = -f(x) \), that is changing the sign of the argument changes the sign of the function. From the figure below it is apparent that a function is odd if it is symmetric through the origin (often referred to as rotational symmetry).

![Figure 5: An odd function.](image)

Can you sketch a function that is neither odd nor even? Is both odd and periodic?
Further descriptors of functions.

A point at which the graph of a function cuts the \( x \) axis is said to be a zero of the function. Thus in the graph of the function

\[
y = (x - 1)(x + 2)
\]

shown below there are zeros at \( x = 1 \) and \( x = -2 \).

A function is said to be increasing over that part of its domain where the graph is rising i.e. for \( y = f(x) \), \( y \) increases as \( x \) increases. In the graph above the function is increasing for \( x > -\frac{1}{2} \).

Similarly a function is said to be decreasing over that part of its domain where its graph is falling i.e. \( y \) decreases as \( x \) increases. In the graph the function is decreasing for \( x < -\frac{1}{2} \).

When a function changes from decreasing to increasing it is said to have a minimum point or simply a minimum (plural minima). In the graph there is a single minimum point at \( x = -\frac{1}{2} \).

Similarly, when a function changes from increasing to decreasing it is said to have a maximum point or simply a maximum (plural maxima). The above graph does not have any maximum points.

Maximum and minimum points are collectively referred to as extremal points or simply extremum (singular extremal point or extrema).

The above generally assumes that the function \( f \) is continuous. (Note that the text varies slightly in some of these definitions.)
**Example** Consider the function $y = 2x^3 + 3x^2 - 12x + 32$ whose graph is shown below.

From the graph we see that the function has a zero at $x = -4$, a maximum point at $x = -2$ and a minimum point at $x = 1$. Also the function is increasing for $-\infty < x < -2$, decreasing for $-2 < x < 1$ and increasing again for $1 < x < \infty$.

![Figure 7: Graph of $y = 2x^3 + 3x^2 - 12x + 32$.](image)