Complex numbers

When solving quadratic equations it is apparent that sometimes a quadratic will have two zeros, sometimes one zero and sometimes no zeros. The Fundamental Theorem of Algebra however states that a polynomial of degree \( n \) will have exactly \( n \) zeros (counting repetitions). Is there some way then of retrieving these ‘missing’ zeros?

Consider the quadratic equation \( x^2 + 2x + 5 \). The usual expression for the zeros gives

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4 \times 5}}{2} = -1 \pm \sqrt{-4}
\]

The problem arises then because we are required to take the square root of a negative number. To retrieve the two zeros we introduce the symbol \( j \) for \( \sqrt{-1} \) (sometimes \( i \) is used instead of \( j \)). For the above quadratic we then have

\[
x = -1 \pm \sqrt{-4} = -1 \pm 2\sqrt{-1} = -1 \pm 2j
\]

and we have the two zeros \( x = -1 + 2j \) and \( x = -1 - 2j \).

We have previously only dealt with real numbers. The present situation, where we also include \( j = \sqrt{-1} \), gives rise to the complex numbers. The general complex number is a number of the form \( x + jy \) where \( x \) and \( y \) are real numbers. Complex numbers are an extension of the real number system in that a real value \( x \) is simply equivalent to the complex value \( x + 0j \).

Complex numbers can be added and subtracted, multiplied and divided in essentially the same way as real numbers except that we need take account of the fact that \( j \times j = -1 \).

Note that the use of complex numbers is not just of theoretical interest since they are a useful device in applications. This is despite the fact that, in a very real sense, they ‘do not exist’.
We review some definitions and the rules of arithmetic for complex numbers.

As stated the general complex number is denoted by \( z = x + jy \) where \( x \) and \( y \) are real numbers. The \textit{real part} of \( z \) (ie \( x \)) is denoted \( \Re(z) \) and the \textit{imaginary part} of \( z \) (ie \( y \)) is denoted \( \Im(z) \). If \( y = 0 \) then \( z \) is said to be \textit{purely real} and if \( x = 0 \) then \( z \) is said to be \textit{purely imaginary}.

The arithmetic of complex numbers is straightforward. In what follows \( z_1 \) and \( z_2 \) are the two complex numbers \( z_1 = x_1 + jy_1, \ z_2 = x_2 + jy_2 \).

- The numbers numbers \( z_1 \) and \( z_2 \) are equal if and only if their real and imaginary parts are equal ie iff \( x_1 = x_2 \) and \( y_1 = y_2 \).
- The sum of the two complex numbers \( z_1 \) and \( z_2 \) is given by \( z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \) ie the real and the imaginary parts are added.
- The difference of \( z_1 - z_2 \) is obtained by forming the difference of the real and imaginary parts.
- The product \( z_1 z_2 \) is obtained by multiplying out the numbers in the usual way, noting that \( j^2 = -1 \):

\[
z_1 \times z_2 = (x_1 + jy_1) \times (x_2 + jy_2) = x_1x_2 + x_1y_2j + jy_1x_2 + j^2y_1y_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2).
\]

- To divide \( z_1 \) by \( z_2 \) we proceed as follows

\[
\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \times \frac{x_2 - jy_2}{x_2 - jy_2} = \frac{x_1x_2 + y_1y_2 + j(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}.
\]
**Example** We demonstrate the rules for complex arithmetic by working a few simple examples.

(i) \((2 - 3j) - 3(1 + j) = (2 - 3j) - (3 + 3j) = -1 - 6j\)

(ii) \((2 - 3j)(1 + j) = 2 + 2j - 3j - 3j^2 = 2 + 2j - 3j + 3 = 5 - j\)

(iii) \(\frac{2 - 3j}{1 + j} = \frac{2 - 3j}{1 + j} \cdot \frac{1 - j}{1 - j} = \frac{(2 - 3j)(1 - j)}{(1 + j)(1 - j)} = \frac{-1 - 5j}{2} = \frac{1}{2} \cdot \frac{5}{2}j\)

A convenient way of representing complex numbers is by identifying the point \((x, y)\) in the cartesian plane with the complex number \(z = x + yj\). In this context the \(x\) axis is referred to as the *real axis*, the \(y\) axis is referred to as the *imaginary axis* and the diagram itself as the *Argand diagram* or simply the *complex plane*. See figure 1.

![Figure 1: The complex plane](image)

Next we define the *complex conjugate* of a complex number \(z = x + yj\) to be the complex number \(\bar{z} = x - yj\) (sometimes denoted \(z^*\)). Thus, in the complex plane, \(\bar{z}\) is \(z\) reflected in the real axis. The following results concerning the complex conjugate are readily verified:

- \(z + \bar{z} = 2x = 2\Re(z)\),
- \(z - \bar{z} = 2jy = 2j\Im(z)\),
- \(z\bar{z} = x^2 + y^2\) ie \(z\bar{z}\) is real.
You are already possibly aware of the alternative representation of points in the $x$-$y$ plane in terms of polar coordinates. It is useful in the present context to represent complex numbers in a form analogous to polar coordinates. Referring to figure 1, we represent $z = x + jy$ by the polar coordinates $r$ and $\theta$ and write $z = r \angle \theta$ where $r$, the distance of $z$ from the origin, is referred to as the modulus of $z$ and $\theta$, the angle the line joining the origin to the point $z$, is the argument of $z$. It is also common to write $r = |z|$ and $\theta = \arg z$.

From figure 1 we see that the values $r$ and $\theta$ are given by $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan \frac{y}{x}$. Clearly the value of $\theta$ is multi valued and it is usual to restrict $\theta$ to its principal value: $-\pi < \arg z \leq \pi$.

**Example**

(i) $3 - 4j = 5 \angle \arctan \frac{-4}{3} = 5 \angle (-0.927)$,
(ii) $-3 - 4j = 5 \angle \arctan \frac{-4}{-3} = 5 \angle -2.214$.

We refer to the complex number $z = x + jy$ as being in cartesian form, while $z = r \angle \theta$ is in polar form. Note that the two forms are related by

$$z = x + jy = r \angle \theta = r (\cos \theta + j \sin \theta)$$

so that $x = r \cos \theta$ and $y = r \sin \theta$.

Suppose that the two numbers $z_1 = r_1 \angle \theta_1$, $z_2 = r_2 \angle \theta_2$ are to be multiplied:

$$z_1 z_2 = r_1 \angle \theta_1 \times r_2 \angle \theta_2$$
$$= r_1 (\cos \theta_1 + j \sin \theta_1) \times r_2 (\cos \theta_2 + j \sin \theta_2)$$
$$= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + j (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2))$$
$$= r_1 r_2 (\cos (\theta_1 + \theta_2) + j \sin (\theta_1 + \theta_2))$$
$$= r_1 r_2 \angle (\theta_1 + \theta_2)$$

That is, when multiplying two complex numbers in polar form, the modulus of the product is the product of the moduli and the argument of the product is the sum of the arguments.

We can also show that, for the division of complex numbers in polar form, that the modulus of the quotient is the ratio of their moduli and the argument of the quotient is the difference of the arguments:

$$z_1 / z_2 = (r_1 / r_2) \angle (\theta_1 - \theta_2).$$

This follows simply if we note $\overline{z_2} = r_2 \angle (-\theta_2)$. 

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We next introduce (without proof) what is known as Euler’s formula:

\[ e^{j\theta} = \cos \theta + j \sin \theta. \]

Noting

\[ z = x + jy = r(\cos \theta + j \sin \theta) = re^{j\theta} \]

we have a third form \( z = re^{j\theta} \) for representing the complex number \( z \). This is the exponential form for \( z \).

*Relationship between the trigonometric and hyperbolic functions*

Starting with Euler’s formula

\[ e^{j\theta} = \cos \theta + j \sin \theta, \]

and the same formula with \( \theta \) replaced by \( -\theta \)

\[ e^{-j\theta} = \cos \theta - j \sin \theta \]

and adding we obtain

\[ e^{j\theta} + e^{-j\theta} = 2 \cos \theta \]

or

\[ \cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}. \]

Similarly, subtracting the original formulae we have

\[ j \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2}. \]

Recalling the definitions of the hyperbolic functions

\[ \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \]

we then have

\[ \cos \theta = \cosh j\theta \quad j \sin \theta = \sinh j\theta. \]
That is, the trigonometric and hyperbolic functions are related by way of complex values for their arguments.

To complete the discussion we note that division of the last two expressions for \( \sin \theta \) and \( \cos \theta \) gives

\[
\tanh j\theta = j \tan \theta
\]

and replacing \( \theta \) by \( j\theta \) in the above expressions we have

\[
\cos j\theta = \cosh \theta, \\
\sin j\theta = j \sinh \theta,
\]

\[
\tan j\theta = j \tanh \theta.
\]

The above results are useful for the derivation of addition formulae, double angle formulae etc previously obtained for the circular functions as well as additional relationships between the trigonometric and hyperbolic functions. Except for the next example which illustrates the process we will not be treating such issues further here.

**Example** If \( z = x + jy \) then we have

\[
\cos z = \cos(x + jy) = \cos x \cos jy - \sin x \sin jy = \cos x \cosh y - j \sin x \sinh y.
\]

Remember here that \( x \) and \( y \) are real. We also note the relationship

\[
e^z = e^{x+jy} = e^x e^{jy} = e^x (\cos y + j \sin y) = e^x \angle y
\]

which has applications in control theory.

To complete our discussions of functions with complex arguments we consider logarithmic functions. We begin with the exponential function \( z = e^w \) with \( z = x + jy \) and \( w = u + jv \). (Remember here that \( x, y, u \) and \( v \) are all real.) Thus \( z = e^w \) becomes

\[
x + jy = e^{u+jv} = e^u e^{jv} = e^u (\cos v + j \sin v).
\]

Equating real and imaginary parts we have

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Thus knowing $u, v$ we have $x, y$. That is knowing $w$ we can calculate $z = e^w$. 
How though, if we know $z$, do we calculate $w$?

We begin with

\[
x^2 + y^2 = (e^u \cos v)^2 + (e^u \sin v)^2 = e^{2u}(\cos^2 v + \sin^2 v) = e^{2u}
\]

so that

\[
e^{2u} = (x^2 + y^2) \text{ or } u = \frac{1}{2} \ln(x^2 + y^2) = \ln \sqrt{x^2 + y^2} = \ln |z|.
\]

Next

\[
\frac{y}{x} = \tan v \text{ or } v = \arctan \frac{y}{x} \text{ or } v = \arg z + 2n\pi
\]

where $n$ is an integer. Thus if $z = e^w$ we define $w = \ln z$ as

\[
\ln z = \ln |z| + j \arg z.
\]