Assignment 2 Solutions

1. Using first the identity \( \sin(A + B) = \cos A \sin B + \sin A \cos B \) with \( A = 2\theta \) and \( B = \pi/3 \) gives

\[
\sin(2\theta + \pi/3) = \cos 2\theta \sin \pi/3 + \sin 2\theta \cos \pi/3 = \frac{\sqrt{3}}{2} \cos 2\theta + \frac{1}{2} \sin 2\theta
\]

(where we have used \( \cos \pi/3 = \frac{1}{2} \) and \( \sin \pi/3 = \frac{\sqrt{3}}{2} \)). Next the identities \( \cos 2A = 2 \cos^2 A - 1 \) and \( \sin 2A = 2 \cos A \sin A \) with \( A = \theta \) give

\[
\sin(2\theta + \pi/3) = \frac{\sqrt{3}}{2} (2 \cos^2 \theta - 1) + \cos \theta \sin \theta
\]

\[
= \sqrt{3} \cos^2 \theta + \cos \theta \sin \theta - \frac{\sqrt{3}}{2}.
\]

Note that alternative rearrangements are possible.

2. Firstly noting \( \sin \theta = 0.4 \) we obtain \( \theta = \sin^{-1} 0.4 = 23.58^\circ \). A second solution is \( \theta = 180 - 23.58 = 156.42^\circ \).

3. We write

\[
3 \sin 3x - 5 \cos 3x = A \sin(3x - \alpha)
\]

\[
= A(\sin 3x \cos \alpha - \cos 3x \sin \alpha)
\]

\[
= A \sin 3x \cos \alpha - A \cos 3x \sin \alpha
\]

Comparing coefficients of \( \cos 3x \) we have \( A \sin \alpha = 5 \) and of \( \sin 3x \) we have \( A \cos \alpha = 3 \). Squaring and adding gives \( A^2 = 34 \) and so we take \( A = \sqrt{34} \approx 5.83 \). Dividing the two equations gives \( \tan \alpha = \frac{\sqrt{34}}{3} \) or \( \alpha = \tan^{-1} \frac{\sqrt{34}}{3} = 59.04^\circ \) or 1.0304 radians. We note that \( \alpha \) is required to lie in the first quadrant as both \( \cos \alpha \) and \( \sin \alpha \) are both positive.

The required expression is thus \( 5.83 \sin(3x - 1.0304) \) in radians or \( 5.83 \sin(3x - 59.04^\circ) \).

4. (i) \( 3a - b = 3(1, 3, -1) - (2, -1, 0) = (3, 9, -3) - (2, -1, 0) = (1, 10, -3) \).

(ii) \( |3a - b| = |(1, 10, -3)| = \sqrt{1 + 100 + 9} = \sqrt{110} \).

(iii) First \( |c| = \sqrt{14} \) so that \( \hat{c} = \frac{1}{\sqrt{14}} (2, -1, 3) \).

5. We here calculate \( |a| = \sqrt{11}, \ |b| = \sqrt{5} \) and \( a \cdot b = 2 - 3 + 0 = -1 \). From

\[
a \cdot b = \frac{\cos \theta}{|a||b|}
\]
(where $\theta$ is the angle between the vectors) we then obtain $\cos \theta = \frac{-1}{\sqrt{55}} \approx -0.1348$ and $\theta = \pi - \cos^{-1} 0.1348 = \pi - 1.4355 = 3.0507$ radians. Alternatively (using in degrees) $\theta = 180 - \cos^{-1} 0.1348 = 180 - 82.25 = 97.75^\circ$.

6. In the usual way we calculate $a \times b$ from the array

$$\begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 3 & -1 \\
2 & -1 & 0
\end{array}$$

and obtain $a \times b = -\mathbf{i} - 2\mathbf{j} - 7\mathbf{k}$.

To show that $a \times b$ is perpendicular to both $a$ and $b$ we merely have to show that $(a \times b) \cdot a$ and $(a \times b) \cdot b$ are each zero. This is easily done:

$$(a \times b) \cdot a = (-1, -2, -7) \cdot (1, 3, -1) = 0$$

$$(a \times b) \cdot b = (-1, -2, -7) \cdot (2, -1, 0) = 0$$

7. The equation of the plane passing through the point with position vector $a$ and having normal $n = (r - a) \cdot n = 0$ or $r \cdot n = a \cdot n$. With $a = (2, 3, -1)$ and $n = (2, 1, -2)$ the equation of the plane is

$$r \cdot (2, 1, -2) = (2, 3, -1) \cdot (2, 1, -2) = 4 + 3 + 2 = 9.$$  

With $r = (x, y, z)$ the equation in cartesian form is $2x + y - 2z = 9$.

8. We here have simply $\sin A = \frac{5}{8}$ or $A = \sin^{-1} \frac{5}{8} = 38.68^\circ$ or $0.6571$ radians.

9. The cosine rule gives (refer to figure 1)

$$QR^2 = PQ^2 + PR^2 - 2PQ \cdot PR \cos P$$

$$= 25 + 36 - 2 \times 5 \times 6 \times \cos 55^\circ = 26.59$$

and $QR = 5.156$. The sine rule now gives

$$\frac{\sin P}{QR} = \frac{\sin Q}{PR} \quad \text{or} \quad \sin Q = \frac{PR}{QR} \cos P$$

so that

$$\sin Q = \frac{6}{5.156} \sin 55^\circ = 0.9532$$

and $Q = 72.41^\circ$. Finally $R = 180 - P - Q = 52.59^\circ$. 

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10. On the first bearing the yacht travels $28 \sin 45^\circ = 19.80$ (nautical) miles east and $28 \cos 45^\circ = 19.80$ miles north (segment $AB$ in figure 2). On the second bearing the yacht travels $30 \sin 45^\circ = 21.21$ miles east and $30 \cos 45^\circ = 21.21$ miles south (segment $BC$). The combined distance travelled is then $19.80 + 21.21 = 41.01$ miles east and $21.21 - 19.80 = 1.41$ miles south from the starting point.

We next calculate the $\theta = \tan^{-1} \frac{1.41}{41.01} = \tan^{-1} 0.0344 = 1.97^\circ$. The bearing is thus $-90 + 1.97 = -88.03$ or $360 - 88.03 = 271.97$. The distance back to the starting point is 41.03 miles and to arrive back at the starting point in five hours requires a speed of 8.21 knots.