TUTORIAL SHEET NUMBER 5

QUESTION 1

(i) \( \lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin(x)} = 0 \), (ii) \( \lim_{x \to a} \frac{\sin(x - a)}{x^2 - a^2} = \frac{1}{2a} \).

(b) Determine the following limits if they exist.

(i) \( \lim_{x \to 0} \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x} \), (ii) \( \lim_{x \to \infty} \frac{(x^2 + (x^2 + 1)^{1/2})^{1/2}}{x} \).

QUESTION 2

(a) Extend the function \( f(x) = \frac{x^2 - 1}{x - 1} \) so that it is continuous everywhere.

(b) Find the following derivatives from first principles

(i) \( f(x) = x^2 \), (ii) \( f(x) = \sqrt{x + 3} \).

QUESTION 3

(a) Show that

Find the derivative of the following functions

(a) \( f(x) = (3x - x^2)(4 + x) \), (b) \( f(x) = \arctan \left( \frac{2x}{1 - x^2} \right) \),

(c) \( f(x) = \exp(5 \cos(x)) - 3x \), (d) \( y = f(x) = \arccos(x) + \frac{x}{1 - x^2} \),

(e) \( f(x) = -\ln \left( \frac{(1 + x^2)^{1/2} + x}{(1 + x^2)^{1/2} - x} \right) \), (f) \( f(x) = \cos(3\ln(x)) \).

QUESTION 4

(a) Use L’Hopital’s Rule (if applicable) to show:
(i) \( \lim_{x \to 0} \frac{\cos(x) + 2x - 1}{3x} = \frac{2}{3} \)  
(ii) \( \lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = 0. \)

(iii) \( \lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6} = \frac{3}{13} \)  
(iv) \( \lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = -\frac{1}{2} \)  
(v) \( \lim_{x \to \infty} \frac{\ln(\ln(x))}{\ln(x)} = 0. \)  
(vi) \( \lim_{x \to 0} \frac{\arcsin(2x)}{\arcsin(x)} = 2. \)

(b) Evaluate the following limits:

(i) \( \lim_{x \to 0^+} x^{1/x} \).

(ii) \( \lim_{x \to 0^+} x^{\sin(x)}. \)

(iii) \( \lim_{x \to \infty} xe^{-\sqrt{x}}. \)

(iv) \( \lim_{x \to 0} \left( 1 + \frac{1}{2x} \right)^{x^2}. \)

(v) \( \lim_{x \to \pi/2} \left( \sec(x) - \tan(x) \right). \)

(vi) \( \lim_{x \to 1^-} \left( \frac{1}{1 - x} - \frac{1}{\ln(x)} \right). \)

QUESTION 5

(a) The function \( f \) can be differentiated three times, and its first derivative is always positive. Two functions \( u \) and \( v \) are defined by the evaluations

\[ u(x) = (f'(x))^{-1/2} \text{ and } v(x) = f(x)u(x). \]

Show that

\[ \frac{v''(x)}{v(x)} = \frac{u''(x)}{u(x)}. \]

Under what conditions on \( x \) does this equation hold?

(b) Find the derivative of the function \( f \) where

\[ y = f(x) = \arctan(1/x). \]

By comparing your answer with the derivative of \( \arctan(x) \), show that

\[ f(x) = \frac{\pi}{2} - \arctan(x) \quad \text{if } x > 0 \]

\[ f(x) = -\frac{\pi}{2} - \arctan(x) \quad \text{if } x < 0. \]

(c) Find the derivative of the function \( g \) where \( g \) is defined by

\[ y = g(x) = \arcsin \left( (1 - x^2)^{-1/2} \right), \]

and give an explanation for your answer.

QUESTION 6

(a) Show that each of the following function has the same derivative in the interval \([\beta, \alpha]::

\[ f_1(x) = 2 \arcsin(\sqrt{|x - \beta|/|\alpha - \beta|}) \]

\[ f_2(x) = 2 \arctan(\sqrt{|x - \beta|/|\alpha - x|}) \]

\[ f_3(x) = \arcsin(2\sqrt{|\alpha - x|/|\alpha - \beta|}) \].
Can you explain why this is so?

(b) Find the angle between the tangents to the curves \(3y = 2x + x^4 y^3\) and \(2y + 3x + y^5 = x^3 y\) at the origin.

(c) Show that
\[
\ln(x) > 2x - x^2/2 - 3/2 \quad \text{whenever } \ x > 1.
\]

Hint: Construct an appropriate function and show by examining its derivative that it is strictly increasing.

**QUESTION 7**

Find the derivative of the following functions.

(a) \(f(x) = x \arctan(x/a) - a \ln(a^2 + x^2)/2\),

(b) \(f(x) = (a^2 + x^2) \arctan(x/a)/2 - ax/2\),

(c) \(f(x) = x(\arcsin(x/a))^2 - 2x + 2(a^2 - x^2)^{1/2} \arcsin(x/2)\),

(d) \(f(x) = x^a\),

(e) \(f(x) = \ln(e^x/(1 + e^x))\),

(f) \(f(x) = e^x(\sin(x) - \cos(x))/(a^2 + 1)\),

(g) \(f(x) = -\ln(x + (x^2 - a^2)^{1/2})/x + \sec^{-1}(|x/a|)/a\).

Here \(a\) is an arbitrary positive real number.

**QUESTION 8**

(a) When a certain bowl is filled with water to a depth of \(x\) cm, the volume of water in the bowl is \(\pi x^2 (6 - x)/3\) cm\(^3\). If the bowl has a hole in the bottom and water is escaping at the constant rate of 3 cm\(^3\)/sec, find the rate at which the depth of water is decreasing at the instant when the depth is 2 cm.

(b) Water is flowing into an upright cylindrical tank at the rate of \(4\pi/5\) cubic metres per minute. The tank is stretching in such a way that, while it remains cylindrical, its radius is increasing at the rate of 0.2 cm/min. How fast is the surface of the water rising when the radius is 2 m and the volume of water in the tank is \(20\pi\) m\(^3\).