SOLUTIONS TO TUTORIAL SHEET 4

QUESTION 1

(a) We are given the matrices:

\[ A = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}. \]

(i) Let \( M(m \times n) \) be the class of all matrices of dimension \( m \times n \). Here \( A \) and \( B \) belong to \( M(2 \times 2) \), that is, \( A, B \in M(2 \times 2) \). Since the number of columns of \( A \) equals the number of rows of \( B \), the product \( AB \) is well defined and we find

\[ AB = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (-1)(1) + (1)(-1) & (-1)(-2) + (1)(1) \\ (3)(1) + (4)(-1) & (3)(-2) + (4)(1) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & -2 \end{bmatrix}. \]

(ii) \( BA \) calculation:

\[ AB = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (1)(-1) + (-2)(3) & (1)(1) + (-2)(4) \\ (-1)(-1) + (1)(3) & (-1)(1) + (1)(4) \end{bmatrix} = \begin{bmatrix} -7 & -5 \\ 4 & 3 \end{bmatrix}. \]

(iii) We note that \( C^T \in M(1 \times 3) \) and \( A \in M(2 \times 2) \). The number of columns of \( C^T \) and the number of rows of \( A \) are not equal so the product \( C^T A \) is not defined.

(iv) \( C C^T \) is a well defined product and \( C C^T \in M(3 \times 3) \). Its evaluation is given as:

\[ CC^T = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix}. \]

(v) \( C^T C \) is a well defined product and \( C^T C \in M(1 \times 1) \), that is, the product is a scalar. It is the scalar product of the vector \( C \) with itself and evaluation is given as:

\[ C^T C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = (1)(1) + (-1)(-1) + (2)(2) = 6. \]

(vi) We show that \((A + B)^T = A^T + B^T\). The result that the transpose of a sum of two matrices of the same dimension equals the sum of the transposes of the two matrices can be shown to be true in general.
\[(A + B)^T = \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] + \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right]^T = \left[ \begin{array}{cc} 0 & -1 \\ 2 & 5 \end{array} \right] = \left[ \begin{array}{cc} 0 & 2 \\ -1 & 5 \end{array} \right].\]

\[
\begin{align*}
A^T + B^T &= \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right]^T + \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right]^T = \left[ \begin{array}{cc} -1 & 3 \\ 1 & 4 \end{array} \right] + \left[ \begin{array}{cc} 1 & -1 \\ -2 & 1 \end{array} \right] = \left[ \begin{array}{cc} 0 & 2 \\ -1 & 5 \end{array} \right] \\
&= (A + B)^T.
\end{align*}
\]

(vii) We show that \((AB)^T = B^T A^T\). The result that the Transpose of a product of two matrices of appropriate dimensions so that the product is well defined equals the reverse product of the transposes of the two matrices can be shown to be true in general.

\[
\begin{align*}
(AB)^T &= \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right]^T = \left[ \begin{array}{cc} -2 & 3 \\ -1 & -2 \end{array} \right] = \left[ \begin{array}{cc} -2 & -1 \\ 3 & -2 \end{array} \right].
\end{align*}
\]

\[
\begin{align*}
B^T A^T &= \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right]^T \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right]^T = \left[ \begin{array}{cc} 1 & -1 \\ -2 & 1 \end{array} \right] \left[ \begin{array}{cc} 1 & 4 \\ 3 & 2 \end{array} \right] \\
&= \left[ \begin{array}{cc} (1)(-1) + (-1)(1) & (1)(3) + (-1)(4) \\ (-2)(-1) + (1)(1) & (-2)(3) + (1)(4) \end{array} \right] = \left[ \begin{array}{cc} -2 & -1 \\ 3 & -2 \end{array} \right] \\
&= (AB)^T.
\end{align*}
\]

(b) We are given the extra matrix \(D \in M(2 \times 2)\) defined by

\[
D = \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right],
\]

and scalars \(a = -3\) and \(b = 2\). It is easy to verify the following results:

(i) \(A + (B + D) = (A + B) + D\). This is the Associative law for addition.

\[
\begin{align*}
A+(B+D) &= \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] + \left( \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right] + \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] \right) = \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] + \left[ \begin{array}{cc} 1 & -3 \\ 3 & 7 \end{array} \right] = \left[ \begin{array}{cc} 0 & -2 \\ 6 & 11 \end{array} \right].
\end{align*}
\]

\[
\begin{align*}
(A+B)+D &= \left( \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] + \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right] \right) + \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] = \left[ \begin{array}{cc} 0 & -1 \\ 2 & 5 \end{array} \right] + \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] = \left[ \begin{array}{cc} 0 & -2 \\ 6 & 11 \end{array} \right].
\end{align*}
\]

(ii) \(A(BD) = (AB)D\). This is the associative law for multiplication.

\[
\begin{align*}
A(BD) &= \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] \left( \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right] \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] \right) = \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] \left[ \begin{array}{cc} -8 & -13 \\ 4 & 7 \end{array} \right] = \left[ \begin{array}{cc} 12 & 20 \\ -8 & -11 \end{array} \right].
\end{align*}
\]

\[
\begin{align*}
(AB)D &= \left( \left[ \begin{array}{cc} -1 & 1 \\ 3 & 4 \end{array} \right] \left[ \begin{array}{cc} 1 & -2 \\ -1 & 1 \end{array} \right] \right) \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] = \left[ \begin{array}{cc} -2 & 3 \\ -1 & -2 \end{array} \right] \left[ \begin{array}{cc} 0 & -1 \\ 4 & 6 \end{array} \right] = \left[ \begin{array}{cc} 12 & 20 \\ -8 & -11 \end{array} \right].
\end{align*}
\]
(iii) \((a + b)D = aD + bD\).

\[
(a + b)D = (-3 + 2)D = (-1) \begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -4 & -6 \end{bmatrix}
\]

\[
aD + bD = -3D + 2D = (-3) \begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} + (2) \begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -12 & -18 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 8 & 12 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -4 & -6 \end{bmatrix} = (a + b)D.
\]

(iv) \(a(B - D) = aB - aD\). (This is easy to establish)

(v) \(a(BD) = (aB)D = B(aD)\). (This is easy to establish)

(vi) \(A(B + D) = AB + AD\). (Distributive law of multiplication over addition)

\[
AB + AD = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 16 & 21 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 7 \\ 16 & 21 \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ 15 & 19 \end{bmatrix} = A(B + D).
\]

(c) We are required to show that for matrices \(A\) and \(B\) in part (a)

\((A + B)^2 \neq A^2 + 2AB + B^2\).

The left hand side equals:

\[
(A + B)^2 = \left( \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 10 & 23 \end{bmatrix}.
\]

The right hand side equals:

\[
A^2 + 2AB + B^2 = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} + 2 \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 9 & 19 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 18 \end{bmatrix}.
\]

The reason that the two evaluations are not equal is that \(AB \neq BA\). Check this.

**QUESTION 2**

(a) Decide whether the following statements are true or false. We are given
First let us assume that matrix $A$ is of size $m \times n$ and $B$ is of size $n \times p$. We write the matrices in the following forms

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} \text{ with product } C = AB = [c_{ij}] = [r_1 b_j].$$

In this notation we have used that $a_i$ is the $i^{th}$ row vector of the matrix $A$ and $c_j$ is the $j^{th}$ column vector of the matrix $B$. Then the $ij^{th}$ of the product $C$ is the scalar product $a_i b_j$. Now to answer the questions.

(i) If the first and third columns of $B$ are the same, so are the first and third columns of $AB$ assuming of course that the product of $A$ and $B$ is well defined.

This statement is TRUE. Now the first and third columns of $C = AB$ are defined by

$$c_1 = \begin{bmatrix} a_1 b_1 \\ a_2 b_1 \\ \vdots \\ a_m b_1 \end{bmatrix} \text{ and } c_3 = \begin{bmatrix} a_1 b_3 \\ a_2 b_3 \\ \vdots \\ a_m b_3 \end{bmatrix}.$$ 

If we are given $b_1 = b_3$, then clearly the first and third columns of the product are equal.

(ii) If the first and third rows of $B$ are the same, so are the first and third rows of $AB$ assuming of course that the product of $A$ and $B$ is well defined.

This statement is FALSE. The following counterexample is given.

$$AB = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}.$$ 

(iii) If the first and third rows of $A$ are the same, so are the first and third rows of $AB$ assuming of course that the product of $A$ and $B$ is well defined.

This statement is TRUE. The first and third rows of the product are given by

$$r_1 = \begin{bmatrix} a_1 b_1 & a_2 b_2 & \cdots & a_1 b_p \end{bmatrix} \text{ and } r_3 = \begin{bmatrix} a_3 b_1 & a_3 b_2 & \cdots & a_3 b_p \end{bmatrix}.$$ 

Clearly these row vectors are equal if $a_1 = a_3$.

(b) We are given the following two matrices:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 2 \end{bmatrix}.$$ 

(i) By direct calculation we verify that $(AB)^T = B^T A^T$.

$$(AB)^T = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & -3 & 4 \\ 3 & 4 & 3 \\ 3 & -5 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 8 \\ 3 & 4 & 3 \\ 4 & -5 & 1 \end{bmatrix}.$$
\[ B^T A^T = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 3 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 2 \\ -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ -13 & 1 & -21 \\ 1 & -1 & 1 \end{bmatrix} = (AB)^T. \]

(ii) We calculate \( AB - BA \).

\[ AB - BA = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ -5 & 4 & -5 \\ 8 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 7 & -1 & 3 \\ -8 & 0 & -2 \\ 6 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -2 & 1 \\ 3 & 4 & -3 \\ 2 & 0 & 0 \end{bmatrix} \]

Since this is not the zero matrix we conclude that \( AB \neq BA \), that is the matrix multiplication for these two matrices is not commutative.

(c) We are given three matrices \( D, E, F \) for which the product \( DEF \) is well defined. Then

\[ (DEF)^T = ((DE)F)^T = F^T (DE)^T = F^T (E^T D^T) = F^T E^T D^T. \]

In this calculation we have used the rule for transposition of two matrices, and removed the brackets in the last expression since matrix multiplication is associative. We can generalise the result to:

\[ (A_1 A_2 \cdots A_{n-1} A_n)^T = A_n^T A_{n-1}^T \cdots A_2^T A_1^T, \]

where it is assumed the product of the \( n \) matrices \( A_k \) on the left is well defined. This result can be proved by induction. You might like to try to prove it.

**QUESTION 3**

(a) A square matrix \( A \) is called symmetric if \( A^T = A \). It is called skew symmetric if \( A^T = -A \). Here \( A^T \) is the transpose of the matrix \( A \), that is, it is the matrix obtained from \( A \) by interchanging the rows and columns of the matrix. The first thing we must not is that if \( A \) is symmetric then by definition

\[ (A^T)^T = A. \]

(i) We are given any matrix \( A \). To prove that \( X = AA^T \) is symmetric we must show that \( X^T = X \). This is shown in the following calculation proving the result.

\[ X^T = (AA^T)^T = (A^T)^T (A)^T = AA^T = X. \]

To prove \( Y = A + A^T \), we must show that \( Y^T = Y \). This is proved as follows:

\[ Y^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = Y; \]

since matrix addition is associative and the transpose of a sum of two matrices is the sum of the transposes of the two matrices.

(ii) To show that \( W = A - A^T \) is skew symmetric, we must show that \( W^T = -W \). The proof is as follows:

\[ W^T = (A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -W. \]
(b) Taking a hint from the previous section we observe that

\[ A = \frac{1}{2} A + \frac{1}{2} A = \frac{A + AT}{2} + \frac{A - AT}{2}. \]

This shows that any square matrix \( A \) can be written as the sum of two matrices, one which is symmetric and the other which is skew symmetric. Why does it not hold if the matrix is not square?

(c) We are given any two matrices \( A \) and \( B \) for which the product \( BA \) is well defined. Then

\[ (A^T B^T)^T = (B^T)^T (A^T)^T = BA. \]

(d) We are given that \( A \) is a square non-singular matrix, that is, it has an inverse \( A^{-1} \) for which \( A A^{-1} = A^{-1} A = I \), where \( I \) is the unit matrix of appropriate dimension. Consider the transposition of \( A A^{-1} \) as shown below:

\[ (A^{-1})^T = A^T (A^{-1})^T = I^T = I. \]

By definition of the inverse of a matrix this equation shows that \( A^T \) is invertible, or non-singular with inverse \( (A^T)^{-1} = (A^{-1})^T \).

**QUESTION 4**

Let \( A \) be a square matrix, \( I \) the corresponding identity matrix of the same dimension and \( O \) the zero matrix of the same dimension.

(a) We are require to show that

\[ X = (I - A)^{-1} = I + A + A^2 + A^3 \text{ if } A^4 = O. \]

Now \( X \) will be the inverse of \( I - A \) if \( (I - A)X = I \), that is,

\[
(I - A)X &= (I - A)(I + A + A^2 + A^3) \\
&= (I + A + A^2 + A^3) - A(I + A + A^2 + A^3) \\
&= I + A + A^2 + A^3 - A - A^2 - A^3 - A^4 \\
&= I - A^4 \\
&= I \text{ since } A^4 = O.
\]

This establishes the required result.

(b) The proof to establish

\[ (I - A)^{-1} = I + A + A^2 + \cdots + A^n \text{ if } A^{n+1} = O, \]

for all \( n \geq 0 \), follows similar to the proof given in (a). Let \( X = I + A + A^2 + \cdots + A^n \). Then for \( n \geq 1 \) (The result is trivially true for \( n = 0 \), verify this),

\[
(I - A)X &= (I - A)(I + A + A^2 + \cdots + A^n) \\
&= (I + A + A^2 + \cdots + A^n) - A(I + A + A^2 + \cdots + A^n) \\
&= (I + A + A^2 + \cdots + A^n) - A - A^2 - A^3 - \cdots - A^n - A^{n+1} \\
&= I - A^{n+1} \\
&= I \text{ since } A^{n+1} = O.
\]
(c) It is easy to verify that the matrix

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \]

satisfies the matrix equation \( A^2 - 2A + I = 0 \). Now observe that

\[ A^2 - 2A + I = 0 \text{ is true if and only if } A(2I - A) = (2I - A)A = I. \]

This shows that \( A \) must be non-singular and that its inverse \( A^{-1} \) is the matrix

\[ A^{-1} = 2I - A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \]

Verify this is correct by calculating \( AA^{-1} \) to show that it equals the unit matrix \( I \).

(c) Consider the set of all \( 2\times2 \) matrices, call it \( M(2\times2) \). If given that the matrix \( B \in M(2\times2) \) satisfies

\[ AB = BA \text{ for ALL matrices } A \in M(2\times2), \]

determine the form of \( B \). Let the unknown matrix \( B \) have the form

\[ B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \]

with entries \( b_k, k = 1, \cdots, 4 \), that we are to determine. Since the result is true for all matrices let us consider matrices

\[ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]

Now \( A_1B = BA_1 \) implies that

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} b_3 & b_4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & b_4 \end{bmatrix}. \]

Now two matrices are equal if and only if their corresponding entries are equal. The above equation implies then that \( b_1 = b_4 \) and \( b_3 = 0 \).

Further \( A_2B = BA_2 \) implies that

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ b_3 & 0 \end{bmatrix}. \]

This matrix equation implies that \( b_2 = 0 \) and also \( b_3 = 0 \). We now have that the matrix \( B \) takes the form

\[ B = \begin{bmatrix} b_4 & 0 \\ 0 & b_4 \end{bmatrix} = b_4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = b_4I. \]

where \( b_4 \) is arbitrary. One can clearly identify then that \( B \) is a multiple of the identity matrix.

**QUESTION 5**

(a) We are given the matrix \( A \).

\[ A = \begin{bmatrix} 1 & 4 & y \\ x & 2 & z \\ -1 & 0 & 3 \end{bmatrix}. \]
Since it is symmetric, then $A = A^T$, that is,

$$A = \begin{bmatrix} 1 & 4 & y \\ x & 2 & z \\ -1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & x & -1 \\ 4 & 2 & 0 \\ y & z & 3 \end{bmatrix} = A^T.$$ 

These matrices are equal if and only if their corresponding entries are equal. This tells us that $x = 4$, $y = -1$ and $z = 0$.

(b) We are given that a square matrix $Q$ is said to be orthogonal if $QQ^T = I$, or equivalently, $Q^{-1} = Q^T$.

(i) Given the matrix $D$

$$D = \begin{bmatrix} 2/\sqrt{5} & d \\ -1/\sqrt{5} & e \end{bmatrix},$$

it is orthogonal if $DD^T = I$, that is,

$$DD^T = \begin{bmatrix} 2/\sqrt{5} & d \\ -1/\sqrt{5} & e \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ d & e \end{bmatrix} = \begin{bmatrix} 4/5 + d^2 & -2/5 + de \\ -2/5 + de & 1/5 + e^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Equating corresponding entries we have the equations

$$4/5 + d^2 = 1, \quad -2/5 + de = 0, \quad \text{and} \quad 1/5 + e^2 = 1.$$

Solving these equations the only possible solutions guaranteed by the second equation, are

$$d = 1/\sqrt{5}, e = 2/\sqrt{5} \quad \text{and} \quad d = -1/\sqrt{5}, e = -2/\sqrt{5}.$$ 

(ii) We are given the matrix $A$,

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix},$$

From (i) select the first solution for $d$ and $e$ then

$$D^T AD = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \text{ by evaluation,}$$

which is a diagonal matrix.

**QUESTION 6**

(a) We are given matrices $A$ and $B$ defined by

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}.$$ 

Their inverses are determined as follows using the following notation for three elementary row operations:

- $R_{ij}(k)$ stands for Row(i) replaced by Row(i) + kRow(j)
- $R_{ij}$ stands for Row(i) and Row(j) are interchanged
- $R_i(k)$ stands for Row(i) replaced by k Row(i)

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where \( \text{Row}(i) \) defines the \( i \)th row of an array. Note how the notation used is placed on the appropriate row to enable clarity and checking of calculations.

\[
\begin{bmatrix}
1 & 3 & 3 & | & 1 & 0 & 0 \\
1 & 3 & 4 & | & 0 & 1 & 0 \\
1 & 4 & 3 & | & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 3 & | & 1 & 0 & 0 \\
0 & 0 & 1 & | & -1 & 1 & 0 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
\end{bmatrix} \quad R_{21}(-1)
\]

\[
\begin{bmatrix}
1 & 3 & 3 & | & 1 & 0 & 0 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
0 & 0 & 1 & | & -1 & 1 & 0 \\
\end{bmatrix} \quad R_{31}(-1)
\]

\[
\begin{bmatrix}
1 & 3 & 3 & | & 1 & 0 & 0 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
0 & 0 & 1 & | & -1 & 1 & 0 \\
\end{bmatrix} \quad R_{23}
\]

\[
\begin{bmatrix}
1 & 3 & 0 & | & 4 & -3 & 0 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
0 & 0 & 1 & | & -1 & 1 & 0 \\
\end{bmatrix} \quad R_{13}(-3)
\]

\[
\begin{bmatrix}
1 & 0 & 0 & | & 7 & -3 & -3 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
0 & 0 & 1 & | & -1 & 1 & 0 \\
\end{bmatrix} \quad R_{12}(-3)
\]

Hence the inverse of \( A \) is

\[
A^{-1} = \begin{bmatrix}
7 & -3 & -3 \\
-1 & 0 & 1 \\
-1 & 1 & 0 \\
\end{bmatrix}
\]

It is easily verified that \( AA^{-1} = I \). Recall the question asked that this be done to check your
Hence the inverse of $B$ is

$$B^{-1} = \begin{bmatrix}
-4 & 3 & 0 & -1 \\
2 & -1 & 0 & 0 \\
-7 & 0 & -1 & 8 \\
6 & 0 & 1 & -7 \\
\end{bmatrix}$$

It is easily verified that $BB^{-1} = I$. Recall the question asked that this be done to check your answer.

(b) The first system of equations can be written in the form

$$Ax = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = b.$$

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Multiplying both sides of this equation by the inverse and using the fact that \(AA^{-1} = I\), where \(I\) is the unit \(3 \times 3\) matrix, the solution is given by

\[
\begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = A^{-1} \begin{pmatrix}
    1 & 0 & 0 \\
    1 & 1 & 0 \\
    -1 & 1 & 0
\end{pmatrix} \begin{pmatrix}
    1 \\
    1 \\
-1
\end{pmatrix} = \begin{pmatrix}
    -3 \\
    -3 \\
    1
\end{pmatrix}
\]

Hence the solution is unique and \(x = 4\), \(y = 0\), and \(z = -1\).

The second system of equations can be written in the form

\[
B \begin{pmatrix}
    x \\
    y \\
    z \\
    w
\end{pmatrix} = \begin{pmatrix}
    6 \\
    1 \\
-3 \\
3
\end{pmatrix}
\]

As described above the solution is given by

\[
\begin{pmatrix}
    x \\
    y \\
    z \\
    w
\end{pmatrix} = B^{-1} \begin{pmatrix}
    6 \\
    1 \\
-3 \\
3
\end{pmatrix} = B^c.
\]

Hence the solution is unique and \(x = -24\), \(y = 11\), \(z = -15\) and \(w = 12\).

(c) Check your answers in (b) by finding the solutions to the systems of equations using Gaussian Elimination.

To solve the system of equations

\[
A \begin{pmatrix}
    x \\
    y \\
    z
\end{pmatrix} = \begin{pmatrix}
    1 \\
    1 \\
1
\end{pmatrix},
\]

by Gaussian Elimination, we must perform the same elementary row operations on the coefficient matrix \(A\), reducing it to upper triangular form. Forming the augmented array

\[
[A | b] = \begin{pmatrix}
    1 & 3 & 3 & 1 \\
    1 & 3 & 4 & 0 \\
1 & 4 & 3 & 1
\end{pmatrix}.
\]
here are the calculations:

\[
\begin{bmatrix}
1 & 3 & 3 & 1 \\
1 & 3 & 4 & 0 \\
1 & 4 & 3 & 1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 3 & 1 \\
0 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 \\
\end{bmatrix} \xRightarrow{R_{21}(-1)}
\begin{bmatrix}
1 & 3 & 3 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix} \xRightarrow{R_{31}(-1)}
\]

With the coefficient matrix now reduced to upper-triangular form, we use back substitution to find the solution. Here \( z = -1, \ y = 0 \) and \( x = -3y - 3z + 1 = 4 \), as we determined previously.

It is left as an exercise to verify the Gaussian elimination for the second system of equations

\[Bx = c.\]

(d) It should be clear that the number of multiplications to get to the answer is much less in (c) than in the calculations in (b). This is the reason that Gaussian Elimination is preferred for computer calculations of systems of equations which involve a large number of variables.

**QUESTION 7**

In providing the solutions to the seven systems of equations in this question, the last augmented array in the Gauss-Jordan reduction is given together with the solution if it exists. Use the results to check your calculations.

(i) Given the system

\[
\begin{align*}
x_1 + x_2 + 2x_3 &= 9 \\
2x_1 + 4x_2 - 3x_3 &= 1 \\
3x_1 + 7x_2 - 5x_3 &= -1 \\
\end{align*}
\]

the final augmented array is

\[
\begin{bmatrix}
1 & 0 & 0 & 5 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

There is a unique solution \( x_1 = 5, \ x_2 = -3 \) and \( x_3 = -1 \).

(ii) Given the system

\[
\begin{align*}
2x_1 + 4x_2 - 3x_3 &= 1 \\
x_1 + x_3 &= 1 \\
-x_1 + 2x_2 - x_3 &= -1 \\
\end{align*}
\]

the final augmented array is

\[
\begin{bmatrix}
1 & 0 & 0 & 4/5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1/5 \\
\end{bmatrix}
\]

There is a unique solution \( x_1 = 4/5, \ x_2 = 0 \) and \( x_3 = 1/5 \).
(iii) Given the system
\[
\begin{align*}
  x_1 - 2x_2 &= 0 \\
  3x_1 + 4x_2 &= -1 \\
  2x_1 - x_2 &= 3
\end{align*}
\]
the final augmented array is
\[
\begin{bmatrix}
  1 & 0 & 2 \\
  0 & 1 & 1 \\
  0 & 0 & -11
\end{bmatrix}.
\]
The last line of the final augmented array shows there cannot be a solution, the system of equations is inconsistent.

(iv) Given the system
\[
2x_1 + 4x_2 - 7x_3 = 8
\]
the final augmented array is
\[
\begin{bmatrix}
  1 & 2 & -7/2 & | & 4
\end{bmatrix}.
\]
The system of equations is consistent there being more variables than the number of equations. It has two free parameters. Let \( x_3 = t \) and \( x_2 = s \) where \( t \) and \( s \) are arbitrary parameters. Then from the last augmented array
\[
x_1 = 4 - 2x_2 + (7/2)x_3 = 4 - 2s + (7/2)t.
\]
Hence the parametric form of the solution is
\[
\begin{align*}
  x_1 &= 4 - 2s + (7/2)t \\
  x_2 &= s \\
  x_3 &= t
\end{align*}
\]
where \( s \) and \( t \) are arbitrary parameters. It may be written in vector form:
\[
\begin{align*}
x &= \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
  4 \\
  0 \\
  0
\end{bmatrix} + s \begin{bmatrix}
  -2 \\
  1 \\
  0
\end{bmatrix} + t \begin{bmatrix}
  7/2 \\
  0 \\
  1
\end{bmatrix} = a + sb + tc,
\end{align*}
\]
where \( a, b \) and \( c \) are appropriately defined. In this form we recognise the solution as the parametric form of the equation of the plane passing the point with position vector \( a \) parallel to the vectors \( b \) and \( c \). This is just another way of interpreting the given system equation, which we know is the cartesian form of this same plane.

(v) Given the system equations
\[
\begin{align*}
  2x_1 + 2x_2 + 2x_3 &= 0 \\
  -2x_1 + 5x_2 + 2x_3 &= 0 \\
  -7x_1 + 7x_2 + x_3 &= 0
\end{align*}
\]
the final augmented array is
\[
\begin{bmatrix}
  1 & 0 & 3/7 & | & 0 \\
  0 & 1 & 4/7 & | & 0 \\
  0 & 0 & 0 & | & 9
\end{bmatrix}.
\]
The last row of zeroes in the final augmented array indicates there are infinitely many solutions. We let \( x_3 = t \) where \( t \) is an arbitrary parameter. Then from the augmented array we determine
\[
\begin{align*}
  x_1 &= -(3/7)t \\
  x_2 &= -(4/7)t \\
  x_3 &= t.
\end{align*}
\]
It may be written in vector form
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -(3/7) \\ -(4/7) \\ 1 \end{bmatrix} = tc,
\]
where \( c = [-3/7 - 4/7 1]^T \). The solution defined parametrically is that of a line, passing through the origin parallel to the vector \( c \).

(vi) Given the system equations
\[
\begin{align*}
  x_1 - 4x_2 + 2x_3 - x_4 &= -1 \\
  2x_1 + x_2 - 2x_3 - 2x_4 &= 2 \\
 -x_1 + 2x_2 - 4x_3 + x_4 &= 1 \\
  3x_1 - 3x_4 &= -3
\end{align*}
\]
the final augmented array is
\[
\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]
The last row of zeros indicates there are infinitely many solutions to the system of equations. We let \( x_4 = t \) where \( t \) is an arbitrary parameter. Hence the parametric form of the solution is
\[
\begin{align*}
  x_1 &= -1 + t \\
  x_2 &= 0 \\
  x_3 &= 0 \\
  x_4 &= t
\end{align*}
\]
where \( t \) is an arbitrary parameter. It may be written in vector form:
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = a + tc,
\]
where \( a \) and \( c \) are defined appropriately.

(vii) Given the system equations
\[
\begin{align*}
  x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
  2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
  5x_3 + 10x_4 + 15x_6 &= 5 \\
  2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{align*}
\]
the final augmented array is
\[
\begin{bmatrix}
1 & 3 & 0 & 4 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

There are infinitely many solutions with three (3) free parameters we let \(x_2 = p, \ x_4 = s\) and
\(x_5 = t\), where \(p, s\) and \(t\) are arbitrary parameters. Hence the parametric form of the solution
is
\[
\begin{align*}
x_1 &= -3p - 4s - 2t \\
x_2 &= p \\
x_3 &= -2s \\
x_4 &= s \\
x_5 &= t \\
x_3 &= 1/3
\end{align*}
\]
where \(p, s\) and \(t\) are arbitrary parameters. It may be written in vector form:
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + p \begin{bmatrix}
-3 \\
1 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
-4 \\
0 \\
1 \\
0
\end{bmatrix} + t \begin{bmatrix}
-2 \\
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6
\end{bmatrix} + \begin{bmatrix}
p \\
s \\
t \\
1/3
\end{bmatrix}
\]
where vectors \(a_1, b, c\) and \(d\) are defined appropriately.

**QUESTION 8**

(i) The two scalar products are calculated as:

(a) \(a \cdot b = (1)(2) + (-2)(-2) + (-1)(-1) = 7\).

(b) \(a \cdot b = (2)(3) + (1)(1) + (1)(-1) = 6\).

(ii) If \(a = a_1\hat{i} + a_2\hat{j} - \hat{k}\) is perpendicular to the vector \(b = 2\hat{i} + \hat{j} - 3\hat{k}\) then
\[
a \cdot b = 2a_1 + a_2 + 3 = 0.
\]
If the projection of \(a\) on \(n = \hat{i} + \hat{j} + 2\hat{k}\) is 12, then \(a \cdot \hat{n} = 12\). Given that \(|n| = \sqrt{6}\) so
\(\hat{n} = n/\sqrt{6}\), this implies that
\[
(a_1 + a_2 - 2)/\sqrt{6} = 12.
\]
We now solve the two linear equations for the unknowns \(a_1\) and \(a_2\), to obtain
\[
a_1 = -(5 + 12\sqrt{6}) \quad \text{and} \quad a_2 = 6 + 24\sqrt{6}.
\]
Hence the required vector
\[
a = -(5 + 12\sqrt{6})\hat{i} + (6 + 24\sqrt{6})\hat{j} - \hat{k}.
\]
(iii) Given that \( x + y + 2k \) is parallel to \( 10i - 5j + k \), then

\[
x + y + 2k = \lambda(10i - 5j + k),
\]

for some scalar \( \lambda \). By equality of vectors, equating components we find that

\[
x = 10\lambda, \quad y = -5\lambda \quad \text{and} \quad 2 = \lambda
\]

Hence \( x = 20 \) and \( y = -10 \). The required vector is \( 20i - 10j + 2k \).

QUESTION 9

Relative to the origin of a reference basis \( \{i, j, k\} \), let \( a = i - j \), \( b = -2i + 4j \) and \( c = i + j + k \), \( n = -2i + j + 4k \), be the position vectors of the points \( A, B \) and \( C \). Further let \( n = -2i + j + 4k \).

The direction of the force \( F \) is given by a unit vector \( \hat{f} \) in the direction from \( B \) to \( C \), that is, \( \hat{f} = (c - b)/|c - b| \). Hence

\[
\hat{f} = (3i - 3j + k)/\sqrt{19},
\]

Now the force has magnitude \( |F| = 10 \) units. The vector representation of the force is therefore

\[
\vec{F} = \frac{10}{\sqrt{19}}(3i - 3j + k).
\]

The component of the force \( F \) in the direction of \( n \) is given by

\[
(F \cdot \hat{n})\hat{n} = \left( F \cdot \frac{n}{|n|} \right) \hat{n} = \left( \frac{10}{\sqrt{19\sqrt{21}}} \right) (-6 + 3 + 4)\hat{n} = \left( \frac{-50}{\sqrt{154\sqrt{21}}} \right) \hat{n} = \left( \frac{-50}{21\sqrt{154}} \right) (-2\hat{i} + \hat{j} + 4\hat{k})
\]

QUESTION 10

Two wires \( AB \) and \( AC \) are attached to the top of the pole as shown in the following Figure 1. We first need the coordinate representation of the vectors to the points \( A, B, \) and \( C \). These are \( a = 4k \), \( b = 2i + 4j \), and \( c = 3i \). Then we can calculate \( \vec{AB} = \vec{b} - \vec{a} = 2i + 4j - 4k \), and \( \vec{AC} = \vec{c} - \vec{a} = 3i - 4k \). The angle \( \theta \) between the lines \( AB \) and \( AC \) can be determined by the scalar product as follows:

\[
\cos(\theta) = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{(2i + 4j - 4k) \cdot (3i - 4k)}{34^{1/2}5} = \frac{22}{34^{1/2}5}.
\]
Solving for $\theta$ we find the angle between the wires at $A$ is $41.01^\circ$.

**QUESTION 11**

We are given $\vec{n} = 2\vec{i} + 5\vec{j} - 3\vec{k}$ and $\vec{a} = -\vec{i} + 2\vec{j} + 2\vec{k}$. Let the position vector to an arbitrary point $P$ on the plane be $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

(i) The perpendicular distance from $O$ to the plane $ON$ is defined by the projection of $\vec{a}$ in the direction of $\vec{ON}$ specified by the unit vector $\hat{n}$; that is, $ON = \vec{a} \cdot \hat{n}$.

As the point $P$ lies on the plane, it too must have the same projection in the direction $\hat{n}$. Hence $\vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n}$, or, multiplying both sides by $|\vec{n}|$, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$. Substituting for the given vectors in this equation we find $2x + 5y - 3z = (-1\vec{i} + 2\vec{j} + 2\vec{k}) \cdot (2\vec{i} + 5\vec{j} - 3\vec{k}) = 2$, as required.

(ii) The perpendicular distance from the origin to the plane is $ON = \vec{a} \cdot \hat{n} = \vec{a} \cdot \vec{n} / |\vec{n}| = 2 / (38)^{1/2}$ (units).

(iii) We evaluate $\vec{OB} \cdot \hat{n}$ to find the projection $OB$ in the direction of the normal. $\vec{OB} \cdot \hat{n} = (\vec{i} - \vec{j} - \vec{k}) \cdot (2\vec{i} + 5\vec{j} - 3\vec{k}) / (38)^{1/2} = 0$. 

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The vector \( \vec{n} = 2\hat{i} + 5\hat{j} - 3\hat{k} \) is perpendicular to a plane which passes through \( A \) with position \( \vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k} \). The projection is zero. Thus \( \vec{OB} \) is parallel to the plane. The perpendicular distance from \( B \) to the plane must also equal \( 2/(38)^{1/2} \) and \( B \) lies on the same side of the plane as the origin. If \( \vec{OB} \cdot \vec{n} > 2/(38)^{1/2} \), then \( B \) would have been on the other side of the plane.

**Figure 2: Figure for Question 4**

**QUESTION 12**

(a) We are to show by vector methods that the diagonals of a parallelogram bisect each other. Without loss of generality we let the parallelogram be defined by the Origin \( O \) and two points \( A \) and \( B \), with position vectors \( \vec{a} \) and \( \vec{b} \) relative to \( O \). Let the remaining corner of the parallel be given by the point \( C \), which has position vector \( \vec{c} = \vec{a} + \vec{b} \) using the parallelogram law for addition of vectors.

Now any point on the line along the diagonal \( OC \) is given parametrically by the equation

\[
\vec{r}_1(t) = t\vec{c}, \quad \text{where } t \text{ is a parameter.}
\]

Any point on the line along the diagonal \( AB \) is given parametrically by the equation

\[
\vec{r}_2(s) = \vec{a} + s(\vec{b} - \vec{a}), \quad \text{where } s \text{ is a parameter.}
\]

These two lines intersect when for some \( s \) and \( t \),

\[
\vec{r}_1(t) = \vec{r}_2(s),
\]

that is,

\[
t\vec{c} = t(a + b) = a + s(b - a).
\]

Rearranging the terms we find

\[
(t - 1 + s)\vec{a} + (t - s)\vec{b} = \vec{c}.
\]

Now vectors \( \vec{a} \) and \( \vec{b} \) are not parallel otherwise a parallelogram would not be defined. This means that \( \vec{b} \) cannot be linear multiple of \( \vec{a} \). Hence the only solution given by this equation is when

\[
t - 1 + s = 0 \quad \text{and} \quad t - s = 0.
\]
These equations have solution \( t = s = 1/2 \). The point of intersection is therefore given by

\[ r_1(1/2) = \frac{1}{2}c = \frac{1}{2}(a + b), \]

that is, the diagonals of the parallelogram must bisect each other.

(b) A four-sided plane figure with all its sides of equal length is called a Rhombus. Suppose one of the corners is the origin \( O \), and the other corners are at points \( A, B \) and \( C \) with position vectors \( a, b \) and \( c = a + b \) relative to \( O \). Since the figure is a Rhombus then \( |a| = |b| \). (This means that \( OA = OB = BC = AC \)). To prove the diagonals are perpendicular, we need to show that with this given condition,

\[ c \cdot (b - a) = 0. \]

Evaluation proceeds as

\[ c \cdot (b - a) = (a + b) \cdot (b - a) \]
\[ = a \cdot b - a \cdot a + b \cdot b - b \cdot a \]
\[ = b \cdot b - a \cdot a \text{ since } a \cdot b = b \cdot a \]
\[ = |b|^2 - |a|^2 \]
\[ = 0 \text{ since } |a| = |b|. \]

**QUESTION 13**

We are given the vectors \( \sim{a} \) and \( \sim{b} \) are defined as follows:

\[ \sim{a} = 3\sim{i} - 4\sim{k}, \text{ and } \sim{b} = 2\sim{i} - 2\sim{j} + \sim{k}. \]

(i) The scalar projection of \( \sim{a} \) on \( \sim{b} \) is defined by \( P = \sim{a} \cdot \hat{\sim{b}} \), where \( \hat{\sim{b}} \) is a unit vector in the direction of the vector \( \sim{b} \). Substituting for the given values

\[ P = \frac{\sim{a} \cdot \sim{b}}{|\sim{b}|} = \frac{(3)(2) + (0)(-2) + (-4)(1)}{5} = \frac{2}{3}. \]

(ii) The required angle \( \theta \) is given by the formula

\[ \cos(\theta) = \frac{\sim{a} \cdot \sim{b}}{|\sim{a}||\sim{b}|} \]

. Evaluation yields

\[ \cos(\theta) = \frac{(3)(2) + (0)(-2) + (-4)(1)}{(5)(3)} = \frac{2}{15} \text{ and } \theta \approx 1^\circ. \]

**QUESTION 14**

(a) We are required to determine a unit vector normal to the plane determined by the points \( (1, 2, -1), (2, 3, 1) \) and \( (3, -1, 2) \). Let \( a = \sim{i} + 2\sim{j} - \sim{k}, b = 2\sim{i} + 3\sim{j} + \sim{k} \) and \( c = 3\sim{i} - \sim{j} + 2\sim{k} \). Two vectors in the plane are

\[ \sim{p}_1 = \sim{b} - \sim{a} = \sim{i} + \sim{j} + 2\sim{k} \text{ and } \sim{p}_2 = \sim{c} - \sim{a} = 2\sim{i} - 3\sim{j} + 3\sim{k}. \]
A vector perpendicular to the plane is
\[ \mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2 = \left| \begin{array}{ccc} i & j & k \\ 1 & 1 & 2 \\ 2 & -3 & 3 \end{array} \right| = \left| \begin{array}{ccc} 1 & 2 & 1 \\ -3 & 3 & 2 \\ 1 & 2 & 3 \end{array} \right| \mathbf{i} + \left| \begin{array}{ccc} 1 & 1 & 2 \\ -3 & 3 & 2 \\ 1 & 2 & 3 \end{array} \right| \mathbf{j} + \left| \begin{array}{ccc} 2 & -3 & 3 \\ -3 & 3 & 2 \\ 1 & 2 & 3 \end{array} \right| \mathbf{k} = 9\mathbf{i} + \mathbf{j} - 5\mathbf{k}. \]

(b) The area of the triangle with vertices at the points defined in part (a) is given by
\[ A = \frac{1}{2} |\mathbf{n}| = \frac{\sqrt{107}}{2} \approx 5.172. \]

(c) We are to determine the cartesian equation of the plane passing through the point \( A = (1, 2, -1) \) and perpendicular to the vector \( \mathbf{n} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \). Let the position vector of \( A \) relative to the origin be \( \mathbf{a} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \), and the position vector to any point \( P \) on the plane be given in cartesian coordinates as \( \mathbf{r} = xi + y\mathbf{j} + zk \).

For the arbitrary point \( P \) to lie on the plane the vector \( \mathbf{r} - \mathbf{a} \) must be perpendicular to vector \( \mathbf{n} \). Hence
\[ (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = (x - 1)(4) + (y - 2)(2) + (z + 1)(-5) = 0, \]
that is,
\[ 4x + 2y - 5z = 13. \]
This is the cartesian equation of the required plane.

(d) We are to determine the normal vector to each of the two planes:
\[ 2x - y + 2z = 1, \text{ and } x - y = 2. \]

As can be seen from the previous derivation the two normals can be taken to be
\[ \mathbf{n}_1 = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{n}_2 = \mathbf{i} - \mathbf{j}. \]

(e) The angle \( \theta \) between the two planes is the angle between the two normals. It is given by
\[ \cos(\theta) = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} \]
. Evaluation yields
\[ \cos(\theta) = \frac{(2)(1) + (-1)(-1) + (2)(0)}{(3)(\sqrt{2})} = 1/\sqrt{2} \text{ and } \theta \approx 45^\circ. \]

(f) Let the vertices be the points \( O = (0, 0, 0), A = (1, 1, 1), B = (1, 2, 1) \) and \( C = (2, 1, 1) \), where \( A, B \) and \( C \) have position vectors \( \mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \) and \( \mathbf{c} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \) respectively.
The volume \( V \) of the tetrahedron \( OABC \) is given by

\[
V = \frac{1}{6} |a \cdot (b \times c)|
\]

Now

\[
b \times c = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 1 \\
2 & 1 & 1
\end{vmatrix} = \hat{i} + \hat{j} - 3\hat{k}.
\]

The volume can now be calculated as

\[
V = \frac{1}{6} |(\hat{i} + \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} - 3\hat{k})| = 1/6.
\]

Since the volume is not zero the three vectors formed by joining the origin to the given three points are not coplanar.

**QUESTION 15**

(i) Relative to the basis \( \{\hat{i}, \hat{j}, \hat{k}\} \) basis,

\[
a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.
\]

Relative to the basis \( \{\hat{i}', \hat{j}', \hat{k}'\} \) basis,

\[
a = a'_1\hat{i}' + a'_2\hat{j}' + a'_3\hat{k}'.
\]

Taking the scalar product of Equation (2) with \( \hat{i}' \) we find

\[
a \cdot \hat{i}' = a'_1(\hat{i}' \cdot \hat{i}') + a'_2(\hat{j}' \cdot \hat{i}') + a'_3(\hat{k}' \cdot \hat{i}') = a'_1,
\]

since \( \{\hat{i}', \hat{j}', \hat{k}'\} \) is an orthonormal basis set. Similarly it is found that

\[
a'_2 = a \cdot \hat{j}' \quad \text{and} \quad a'_3 = a \cdot \hat{k}'.
\]

Let us use these 3 equations and substitute for \( a \) from Equation (1). Then

\[
a'_1 = a \cdot \hat{i}' = a_1(\hat{i} \cdot \hat{i}') + a_2(\hat{j} \cdot \hat{i}') + a_3(\hat{k} \cdot \hat{i}')
\]

\[
a'_2 = a \cdot \hat{j}' = a_1(\hat{i} \cdot \hat{j}') + a_2(\hat{j} \cdot \hat{j}') + a_3(\hat{k} \cdot \hat{j}')
\]

\[
a'_3 = a \cdot \hat{k}' = a_1(\hat{i} \cdot \hat{k}') + a_2(\hat{j} \cdot \hat{k}') + a_3(\hat{k} \cdot \hat{k}')
\]

Equations (3), (4) and (5) relate the new coefficients \( a'_k \) to the old \( a_k \) provided the coefficients bracketed can be evaluated given the information on the new basis relative to the old basis.

(ii) Given \( a = 3\hat{i} + 2\hat{j} + \hat{k} \), then \( a_1 = 3, a_2 = 2 \) and \( a_3 = 1 \). By definition the unit vector \( \hat{i}' \) in the direction \( OX' \) is

\[
\hat{i}' = \cos(45^\circ)\hat{i} + \sin(45^\circ)\hat{j} = (\hat{i} + \hat{j})/\sqrt{2}.
\]

Also we find

\[
\hat{j}' = \cos(45^\circ)\hat{j} - \sin(45^\circ)\hat{i} = (\hat{j} - \hat{i})/\sqrt{2},
\]
The coefficients of Equations (3), (4) and (5) are calculated as:

\[ i \cdot i' = \frac{1}{\sqrt{2}} \quad j \cdot j' = \frac{1}{\sqrt{2}} \quad k \cdot k' = 0. \]

\[ i \cdot j' = -\frac{1}{\sqrt{2}} \quad j \cdot i' = \frac{1}{\sqrt{2}} \quad k \cdot j' = 0. \]

\[ i \cdot k' = 0 \quad j \cdot i' = 0 \quad k \cdot i' = 1. \]

Substituting for these values and \( a_k \) into (3), (4) and (5), we find

\[ a_1' = \frac{5}{\sqrt{2}} \quad a_2' = -\frac{1}{\sqrt{2}} \quad \text{and} \quad a_3' = 1. \]