SOLUTIONS TO TUTORIAL 6

6.1 Logarithmic Differentiation

Question 1.

i) The first step in logarithmic differentiation is to take the log of both sides of the equation. However a little bit of care needs to be taken with this step. Recall that you cannot take a log of a negative number. Here we have the equation

$$y = (2x + 1)^5(x^4 - 3)^6.$$

Even powers will always yield positive numbers, however for certain values of $x$, $(2x + 1)^5$ will be negative (and thus $y$ will also be negative). Consequently, taking logs whilst the equation in this form is not valid. To resolve this, we first take the absolute value of both sides of the equation, ensuring that values on both sides of the equation are always positive for any $x$ we wish to consider.

$$|y| = |2x + 1|^5|x^4 - 3|^6.$$

Proceeding with applying logs and the log rules to both sides of the equation we have,

$$\ln |y| = \ln (|2x + 1|^5|x^4 - 3|^6),$$

$$\ln |y| = \ln (|2x + 1|^5) + \ln (|x^4 - 3|^6),$$

$$\ln |y| = 5 \ln (|2x + 1|) + 6 \ln (|x^4 - 3|).$$

Taking the derivative of both sides of the equation,

$$\frac{d}{dx} (\ln |y|) = 5 \frac{d}{dx} (\ln (|2x + 1|)) + 6 \frac{d}{dx} (\ln (|x^4 - 3|)),$$

$$\frac{dy}{dy} (\ln |y|) \frac{dy}{dx} = 5 \frac{d}{du} (\ln (|u|)) \frac{du}{dx} + 6 \frac{d}{dv} (\ln (|v|)) \frac{dv}{dx},$$
where \( u = 2x + 1, \) and \( v = x^4 - 3. \) For these substitutions we have \( du/dx = 2 \) and \( dv/dx = 4x^3. \) Then

\[
\frac{1}{y} \frac{dy}{dx} = (5) \left( \frac{1}{u} \right) (2) + (6) \left( \frac{1}{v} \right) (4x^3),
\]

\[
\frac{1}{y} \frac{dy}{dx} = \left( \frac{10}{2x+1} \right) + \left( \frac{24x^3}{x^4-3} \right).
\]

Therefore

\[
\frac{dy}{dx} = y \left[ \left( \frac{10}{2x+1} \right) + \left( \frac{24x^3}{x^4-3} \right) \right].
\]

ii) Taking logs of both sides of the equation and applying log rules,

\[
\ln (x^y) = \ln (y^x),
\]

\[
y \ln (x) = x \ln (y).
\]

Taking the derivative and applying the product rule on either side of the equation,

\[
\frac{d}{dx} [y \ln(x)] = \frac{d}{dx} [x \ln(y)],
\]

\[
y \frac{d}{dx} [\ln(x)] + \ln(x) \frac{dy}{dx} = x \frac{d}{dx} [\ln(y)] + \ln(y) \frac{d}{dx} [x],
\]

\[
(\ln(x) + x) \frac{dy}{dx} = x \frac{d}{dy} [\ln(y)] \frac{dy}{dx} + \ln(y),
\]

\[
(\ln(x) + x) \frac{dy}{dx} = (x/y) \frac{dy}{dx} + \ln(y).
\]

Re-arranging,

\[
\frac{dy}{dx} = \frac{\ln(y) - y/x}{\ln(x) - x/y}.
\]

**Question 2.** The volume of a cube is given by the formula

\[V(s) = s^3,\]

where \( s \) is the length of one of the sides. When \( s = 30 \) the volume will be \( 27000 \) \( cm^3. \)

Finding the differential,

\[
dV = \frac{dV}{ds} ds,
\]

\[
= 3s^2 ds.
\]
Here we are given a possible error in \( s \) of 0.1\( cm \), consequently, we set \( ds = 0.1 \). The linear change (error) in the volume given a change \( ds \) when \( s = 30 cm \) is then

\[
dV = 3(30)^2(0.1),
\]
\[
= 270 cm^3.
\]

The relative error is

\[
\text{rel. error} = \frac{270}{27000},
\]
\[
= 0.01,
\]

and percentage error

\[
\% \text{ error} = \text{rel.error}(100),
\]
\[
= 1\%.
\]