SOLUTIONS TO TUTORIAL SHEET 5

QUESTION 1

(a) (i) Write the limit as

\[ L = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin(x)} = \lim_{x \to 0} \frac{x \sin(1/x)}{(\sin(x)/x)} = \lim_{x \to 0} \frac{f(x)}{g(x)}. \]

We know that (can be proved using L’Hopital’s rule)

\[ \lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin(x)}{x} = 1. \]

Hence by the quotient rule for limits:

\[ L = \lim_{x \to 0} \frac{x \sin(1/x)}{\sin(x)/x}, \]

provided the limit on the numerator exists. Since \(|\sin(w)| \leq 1\) for all real \(w\), we know that

\[ 0 \leq |x \sin(1/x)| \leq |x|. \]

Now by the squeeze principle, or the pinching theorem, since

\[ \lim_{x \to 0} |x| = 0, \text{ then } \lim_{x \to 0} x \sin(1/x) = 0. \]

We conclude then that the required limit \(L = 0\), as required.

You should be aware that it is \textit{NOT possible} to use L’Hopital’s rule here. Why? The reason is that the following limit

\[ \lim_{x \to 0} f’(x) = \lim_{x \to 0} [2x \sin(1/x) + (x^2) \cos(1/x)[-1/x^2]] = \lim_{x \to 0} (2x \sin(1/x) - \cos(1/x)) \]

is indeterminate.

(ii) We are required to show that

\[ L = \lim_{x \to a} \frac{\sin(x - a)}{x^2 - a^2} = \frac{1}{2a}. \]
The result can be obtained as follows:
\[
L = \lim_{x \to a} \frac{\sin(x - a)}{x^2 - a^2} = \lim_{x \to a} \frac{\sin(x - a)}{(x - a)(x + a)} = \lim_{h \to 0} \frac{\sin(h)}{h((2a + h)^2)}
\]
where we have defined \( h = x - a \) in the limit process. Using the product rule for limits
\[
L = \lim_{h \to 0} \frac{\sin(h)}{h((2a + h)^2)} = \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{1}{(2a + h)^2} = \frac{1}{2a}.
\]

(b) (i) Let the limit
\[
L = \lim_{x \to 0} \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x}.
\]
We could use L’Hopital’s rule here but it is more worthwhile to first rationalise the expression as follows and use the limit theorems:
\[
L = \lim_{x \to 0} \frac{(1 + x)^{1/2} - (1 - x)^{1/2}}{x} \cdot \frac{(1 + x)^{1/2} + (1 - x)^{1/2}}{(1 + x)^{1/2} + (1 - x)^{1/2}}
\]
\[
= \lim_{x \to 0} \frac{((1 + x)^{1/2} - (1 - x)^{1/2})((1 + x)^{1/2} + (1 - x)^{1/2})}{x((1 + x)^{1/2} + (1 - x)^{1/2})}
\]
\[
= \lim_{x \to 0} \frac{2x}{2} = \lim_{x \to 0} \frac{2}{((1 + x)^{1/2} + (1 - x)^{1/2})} = 1.
\]

(ii) Let the limit
\[
L = \lim_{x \to \infty} \frac{(x^2 + (x^2 + 1)^{1/2})^{1/2}}{x}.
\]
We first note that for all real \( x \), in general,
\[
(x^2)^{1/2} = |x| \quad \text{and} \quad (x^2)^{1/2} = x, \quad \text{if} \ x \geq 0.
\]
Then
\[
\frac{(x^2 + (x^2 + 1)^{1/2})^{1/2}}{x} = \frac{x(1 + \frac{1}{x^2})(1 + \frac{1}{x^2})^{1/2}}{x} = (1 + \frac{1}{x}(1 + \frac{1}{x^2})^{1/2})^{1/2}.
\]
Hence
\[
L = \lim_{x \to \infty} (1 + \frac{1}{x}(1 + \frac{1}{x^2})^{1/2})^{1/2} = 1.
\]

**QUESTION 2**

(a) We are given the function
\[
f(x) = \frac{(x^2 - 1)}{(x - 1)}.
\]
\( f(x) \) is valid and defined everywhere except for \( x = 1 \) (cannot divide by zero). Also, for any \( x \neq 1 \) we can use the difference of squares to simplify the expression for \( f(x) \):
\[
f(x) = \frac{(x^2 - 1)}{(x - 1)} = \frac{(x - 1)(x + 1)}{(x - 1)} = x + 1 \quad \text{for} \ x \neq 1.
\]
Thus $f(x)$ is simply the straight line $x + 1$ with a discontinuity - i.e. a hole at $x = 1$. To continuously extend $f(x)$ all we need to do is 'plug' this hole. Formally

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x^2 - 1)}{(x - 1)},$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)},$$

$$= \lim_{x \to 1} (x + 1),$$

$$= 2.$$

This is consistent with substituting $x = 1$ into the equation $x + 1$. Thus a suitable, continuous extension for $f$ is simply the function $g$ defined by.

$$g(x) = x + 1.$$

(b) Solving from first principles requires us to go back to the limit definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

(i) $f(x) = x^2$. 

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h},$$

$$= \lim_{h \to 0} \frac{(x + h - x)(x + h + x)}{h},$$

$$= \lim_{h \to 0} \frac{h(x + h + x)}{h},$$

$$= \lim_{h \to 0} (x + h + x),$$

$$= 2x.$$

Note that we have used the difference of squares to factorize the numerator.

(ii) $f(x) = \sqrt{x + 3}$.

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

$$= \lim_{h \to 0} \frac{\sqrt{x + 3 + h} - \sqrt{x + 3}}{h},$$

$$= \lim_{h \to 0} \frac{\sqrt{x + 3 + h} - \sqrt{x + 3}}{h} \left( \frac{\sqrt{x + 3 + h} + \sqrt{x + 3}}{\sqrt{x + 3 + h} + \sqrt{x + 3}} \right),$$

$$= \lim_{h \to 0} \frac{(x + 3 + h) - (x + 3)}{h(\sqrt{x + 3 + h} + \sqrt{x + 3})},$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x + 3 + h} + \sqrt{x + 3})},$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x + 3 + h} + \sqrt{x + 3}}.$$
\[
\lim_{h \to 0} \frac{h}{h(\sqrt{x + 3} + h + \sqrt{x + 3})} = \frac{1}{2\sqrt{x + 3}}.
\]

**QUESTION 3**

(a) Given the function \( f \) defined by \( f(x) = (3x - x^2)(4 + x) \), then by the product rule

\[ f'(x) = [3 - 2x](4 + x) + (3x - x^2)[1] = 12 - 2x - 3x^2. \]

(b) Given the function \( f \) defined by

\[ f(x) = \arctan \left( \frac{2x}{1 - x^2} \right), \]

we find the derivative by application of the chain rule. Write

\[ y = (F \circ h)(x) = F(h(x)) = F(w) = \arctan(w) \quad \text{where} \quad w = h(x) = \frac{2x}{1 - x^2}. \]

Now by the quotient rule

\[
\frac{dw}{dx} = h'(x) = \frac{2(1 - x^2) + 4x^2}{(1 - x^2)^2}
\]

\[
= \frac{2(1 + x^2)}{(1 - x^2)^2}
\]

Then by the chain rule

\[
f'(x) = \frac{d(\arctan(w))}{dw} \frac{dw}{dx} = F'(w)h'(x) \quad \text{where} \quad w = h(x)
\]

\[
= \frac{1}{1 + w^2} \frac{2(1 + x^2)}{(1 - x^2)^2} \quad \text{where} \quad w = h(x)
\]

\[
= \frac{2}{1 + x^2} \quad \text{after some simplification.}
\]

(c) We are given the function \( f \) defined by

\[ f(x) = \exp(5 \cos(x)) - 3x = g(x) - 3x, \quad \text{where} \quad g(x) = \exp(5 \cos(x)). \]

We first find the derivative of \( g \). Write

\[ g(x) = (G \circ h)(x) = G(h(x)) = G(w) = \exp(w) \quad \text{where} \quad w = h(x) = 5 \cos(x). \]

Then by the chain rule

\[
g'(x) = \frac{d(\exp(w))}{dw} \frac{dw}{dx} = G'(w)h'(x) \quad \text{where} \quad w = h(x)
\]

\[
= \exp(w)[-5 \sin(x)] \quad \text{where} \quad w = h(x)
\]

\[
= -5 \sin(x) \exp(5 \cos(x)).
\]
Hence the derivative of \( f \) is:

\[
f'(x) = g'(x) - 3 = -5 \sin(x) \exp(5 \cos(x)) - 3.
\]

(d) We are given the function \( f \) defined by

\[
f(x) = \arccos(x) + \frac{x}{1 - x^2}.
\]

Then using the quotient rule for the last term

\[
f'(x) = -\frac{1}{(1 - x^2)^{1/2}} + \frac{[1](1 - x^2) - x[-2x]}{(1 - x^2)^2} = -\frac{1}{(1 - x^2)^{1/2}} + \frac{1 + x^2}{(1 - x^2)^2}.
\]

(e) We are given the function \( f \) defined by

\[
f(x) = -\ln \left( \frac{(1 + x^2)^{1/2} + x}{(1 + x^2)^{1/2} - x} \right).
\]

It is best in this question to rationalise the argument of the logarithm function first. Here

\[
\frac{(1 + x^2)^{1/2} + x}{(1 + x^2)^{1/2} - x} = \frac{((1 + x^2)^{1/2} + x)((1 + x^2)^{1/2} + x)}{((1 + x^2)^{1/2} - x)((1 + x^2)^{1/2} + x)} = \frac{((1 + x^2)^{1/2} + x)^2}{((1 + x^2) - x^2)} = \frac{((1 + x^2)^{1/2} + x)^2}{(1 + x^2)(x^2)}.
\]

Hence

\[
f(x) = -\ln(((1 + x^2)^{1/2} + x)^2) = -2 \ln((1 + x^2)^{1/2} + x).
\]

We can now use the chain rule to find the derivative of \( f \). Write

\[
f(x) = (F \circ h)(x) = F(h(x)) = F(w) = -2 \ln(w) \quad \text{where} \quad w = h(x) = (1 + x^2)^{1/2} + x.
\]

Then also using the chain rule to find \( h'(x) \), we evaluate \( f'(x) \) as follows:

\[
f'(x) = \frac{d(-2\ln(w))}{dw} \frac{dw}{dx} = F'(w)h'(x) \quad \text{where} \quad w = h(x)
\]

\[
= -\frac{2}{w} \left[ \frac{1}{2}(2x)(1 + x^2)^{-1/2} + 1 \right] \quad \text{where} \quad w = h(x)
\]

\[
= -\frac{2}{w} \left[ \frac{x}{(1 + x^2)^{1/2}} + 1 \right] \quad \text{where} \quad w = h(x)
\]

\[
= -\frac{2}{(x + (1 + x^2)^{1/2})} \frac{(x + (1 + x^2)^{1/2})}{(1 + x^2)^{1/2}} = -\frac{2}{(1 + x^2)^{1/2}}.
\]
(f) We are given the function \( f \) defined by \( f(x) = \cos(3 \ln(x)) \). It can be written as the composition of two functions as follows
\[
f(x) = (F \circ h)(x) = F(h(x)) = F(w) = \cos(w) \text{ where } w = h(x) = 3 \ln(x).
\]

Then the derivative of \( f \) is calculated as
\[
f'(x) = \frac{d(\cos(w))}{dw} \frac{dw}{dx} = F'(w)h'(x) \text{ where } w = h(x)
\]
\[
= -\sin(w) \left[ \frac{3}{x} \right] \text{ where } w = h(x)
\]
\[
= -\frac{3\sin(3\ln(x))}{x}.
\]

**QUESTION 4**

(a) We evaluate the limits as follows:

(i) Defining \( f \) and \( g \) by \( f(x) = \cos(x) + 2x - 1 \) and \( g(x) = 3x \), we can write the limit as
\[
\lim_{x \to 0} \frac{\cos(x) + 2x - 1}{3x} = \lim_{x \to 0} \frac{f(x)}{g(x)}
\]

It is clear in this case by the continuity of \( f \) and \( g \) that \( f(0) = 0 \) and \( g(0) = 0 \). Then by L’Hôpital’s rule:
\[
\lim_{x \to 0} \frac{\cos(x) + 2x - 1}{3x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\sin(x) + 2}{3} = \frac{2}{3} \text{ as required.}
\]

(ii) Defining \( f \) and \( g \) by \( f(x) = \ln(x) \) and \( g(x) = \sqrt{x} \), we can write the limit as
\[
\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{f(x)}{g(x)}
\]

It is clear in this case
\[
\lim_{x \to \infty} f(x) = \infty \text{ and } \lim_{x \to \infty} g(x) = \infty.
\]

Then by L’Hôpital’s rule:
\[
\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x \to \infty} \frac{2}{x^{1/2}} = 0 \text{ as required.}
\]

(iii) In this problem it is better to check the factorisation of denominator and numerator for a common factor. We find
\[
\lim_{x \to 2} \frac{2x^2 - 5x + 2}{5x^2 - 7x - 6} = \lim_{x \to 2} \frac{(2x - 1)(x - 2)}{(5x + 3)(x - 2)} = \lim_{x \to 2} \frac{(2x - 1)}{(5x + 3)} = \frac{3}{13} \text{ by limit quotient rule.}
\]
Cancellation of the factor \((x - 2)\) can take place since in the limit process \(x \neq 2\). The limit could have also been obtained using L'Hopital's rule.

(iv) Defining \(f\) and \(g\) by \(f(x) = \sin(x) - x\) and \(g(x) = \tan(x) - x\), we can write the limit as

\[
\lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = \lim_{x \to 0} \frac{f(x)}{g(x)}
\]

It is clear in this case by continuity of \(f\) and \(g\)

\[
\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} g(x) = 0.
\]

Then by L'Hopital's rule:

\[
\lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos(x) - 1}{\sec^2(x) - 1}
\]

In this case by continuity of \(f'\) and \(g'\)

\[
\lim_{x \to 0} f'(x) = 0 \quad \text{and} \quad \lim_{x \to 0} g'(x) = 0,
\]

so we need to apply L'Hopital's rule again to obtain:

\[
\lim_{x \to 0} \frac{\sin(x) - x}{\tan(x) - x} = \lim_{x \to 0} \frac{-\sin(x)}{2\sec(x)\sec(x)\tan(x)} = \lim_{x \to 0} \frac{-\cos^3(x)}{2} = \lim_{x \to 0} \frac{-1}{2} \quad \text{by the quotient limit rule}
\]

(v) Defining \(f\) and \(g\) by \(f(x) = \ln(\ln(x))\) and \(g(x) = \ln(x)\), we can write the limit as

\[
\lim_{x \to \infty} \frac{\ln(\ln(x))}{\ln(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)}
\]

It is clear in this case

\[
\lim_{x \to \infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} g(x) = \infty.
\]

Then by L'Hopital's rule:

\[
\lim_{x \to \infty} \frac{\ln(\ln(x))}{\ln(x)} = \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1}{g'(x)} = \lim_{x \to \infty} \frac{1/\ln(1/x)}{(1/x)}
\]

\[
= \lim_{x \to \infty} \frac{1}{\ln(x)} = 0 \quad \text{as required.}
\]

(vi) Defining \(f\) and \(g\) by \(f(x) = \arcsin(2x)\) and \(g(x) = \arcsin(x)\), we can write the limit as

\[
\lim_{x \to 0} \frac{\arcsin(2x)}{\arcsin(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)}
\]

It is clear in this case by continuity of \(f\) and \(g\)

\[
\lim_{x \to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} g(x) = 0.
\]
Then by L'Hopital's rule:
\[
\lim_{x \to 0} \frac{\arcsin(2x)}{\arcsin(x)} = \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2/(1 - (2x)^2)^{1/2}}{1/(1 - x^2)^{1/2}} = \lim_{x \to 0} \frac{2(1 - x^2)^{1/2}}{(1 - x^2)^{1/2}} = 2.
\]

(b) In more brief notation we evaluate each of the limits as follows:

(i)
\[
\lim_{x \to 0^+} \frac{-x}{\ln(x)} = \lim_{x \to 0^+} \frac{-1}{1/x} = \lim_{x \to 0^+} \frac{-1}{\ln(x)} \quad (\text{L'Hopital's rule}) = 0.
\]

So
\[
\lim_{x \to 0^+} \frac{-1}{\ln(x)} = \infty \quad \text{and} \quad \lim_{x \to 0^+} x^{-1/x} = \lim_{x \to 0^+} e^{-(1/x)\ln(x)} = \infty.
\]

(ii)
\[
\lim_{x \to 0^+} x \sin(x) \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/\sin(x)} = \lim_{x \to 0^+} \frac{1/x}{-\cot(x)/\sin(x)} = \lim_{x \to 0^+} \frac{\sin(x)}{x \cos(x)} = (1)(0) = 0.
\]

Hence
\[
\lim_{x \to 0^+} x^2 \sin(x) = \lim_{x \to 0^+} e^{\sin(x) \ln(x)} = e^0 = 1,
\]

using the continuity of the natural logarithm at \( x = 0 \).

(iii)
\[
\lim_{x \to \infty} xe^\sqrt{x} = \lim_{t \to \infty} \frac{t^2}{e^t} \quad (\text{where } t = \sqrt{x}) = \lim_{t \to \infty} \frac{2t}{e^t} = \lim_{t \to \infty} \frac{2}{e^t} = \lim_{t \to \infty} \frac{2}{e^t} \quad (\text{L'Hopital's rule}) = 0.
\]

(iv)
\[
\lim_{x \to \infty} x^2 \ln \left(1 + \frac{1}{2x}\right) = \lim_{t \to \infty} \frac{\ln(1 + \frac{1}{2t})}{1/x^2} = \lim_{t \to \infty} \frac{\left(\frac{-1}{2t}\right)\left(\frac{-1}{2t}\right)}{-2/x^3} = \lim_{x \to \infty} \frac{x}{(2)(2)(1 + \frac{1}{2x})} = \infty.
\]

Hence
\[
\lim_{x \to \infty} \frac{1}{x^2} \ln(1 + \frac{1}{2x}) = \lim_{x \to \infty} e^{x^2 \ln(1 + \frac{1}{2x})} = \infty.
\]

(v)
\[
\lim_{x \to \pi/2} \sec(x) - \tan(x) = \lim_{x \to \pi/2} \frac{1 - \sin(x)}{\cos(x)} = \lim_{x \to \pi/2} \frac{-\cos(x)}{\cos(x)} \quad (\text{L'Hopital's rule}) = 0.
\]

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(vi)
\[
\lim_{x \to \pi/2} \frac{1}{1 - x} - \frac{1}{\ln(x)} = \lim_{x \to \pi/2} \frac{\ln(x) - 1 + x}{(1 - x)\ln(x)} = \lim_{x \to \pi/2} \frac{\frac{1}{x} + 1}{1 - \ln(x)} (\text{L'Hôpital's rule}) = \infty.
\]

**QUESTION 5**

(a) We are given the two functions \( u \) and \( v \) are defined by the evaluations

\[
u(x) = (f'(x))^{-1/2} \quad \text{and} \quad v(x) = f(x)u(x),
\]

where the function \( f \) can be differentiated three times, and its first derivative is always positive. Using the product rule twice

\[
u''(x) = f''(x)u(x) + 2f'(x)u'(x) + f(x)u''(x).
\]

Given \( f'(x) > 0 \) and \( u \) is defined by the square root function, its values must always be positive. On the assumption that \( f(x) > 0 \), then \( v(x) > 0 \), we can form \( v''(x)/v(x) \) and obtain

\[
\frac{v''(x)}{v(x)} = \frac{f''(x)}{f(x)} + 2f'(x)\frac{u'(x)}{u(x)} + \frac{u''(x)}{u(x)}.
\]

Now using the chain rule

\[
u'(x) = -\frac{1}{2} f''(x) (f'(x))^{3/2}, \quad \text{and} \quad \frac{u'(x)}{u(x)} = -\frac{1}{2} (f'(x))^{3/2} (f''(x))^{-1/2} = -\frac{1}{2} f''(x).
\]

Hence by substitution we obtain

\[
\frac{v''(x)}{v(x)} = \frac{u''(x)}{u(x)}.
\]

Clearly the values of \( x \) for which this equation holds are those for which \( f(x) > 0 \).

(b) We are given the function function \( f \) defined by its evaluation

\[
y = f(x) = \arctan(1/x).
\]

Writing \( f \) as the composition \( F \circ h \),

\[
y = f(x) = (F \circ h)(x) = F(h(x)) = F(w) = \arctan(w) \quad \text{where} \quad w = h(x) = 1/x,
\]

we find its derivative to be

\[
f'(x) = \frac{F}{dw} \frac{dw}{dx} = f'(w)h'(x) \quad \text{where} \quad w = h(x) = 1/x
\]

\[
= \frac{1}{1 + w^2} \left[ -\frac{1}{x^2} \right] \quad \text{where} \quad w = h(x) = 1/x
\]

\[
= \frac{1}{1 + 1/x^2} \left[ -\frac{1}{x^2} \right] = -\frac{1}{1 + 1/x^2}
\]

Now we know that if \( g \) is defined by

\[
g(x) = \arctan(x), \quad \text{then} \quad g'(x) = \frac{1}{1 + x^2}.
\]
Hence it is clear that
\[
\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) = 0,
\]
and that
\[
f(x) + g(x) = f(x) + \arctan(x) = C, \quad \text{where } C \text{ is a constant.}
\]
Now \(f\) is not defined for \(x = 0\), but it is defined and continuous for all other real values of \(x\). The constant \(C\) may thus take different values for \(x > 0\) and \(x < 0\).

If \(x > 0\), take \(x = 1\). then \(f(1) = g(1) = \arctan(1) = \pi/4\). Hence \(C = \pi/2\).

If \(x < 0\), take \(x = -1\). then \(f(-1) = g(-1) = \arctan(-1) = -\pi/4\). Hence \(C = -\pi/2\).

This shows as required, that
\[
f(x) = \begin{cases} 
\pi/2 - \arctan(x) & \text{if } x > 0 \\
-\pi/2 - \arctan(x) & \text{if } x < 0.
\end{cases}
\]

(c) We are given the function \(g\) where \(g\) is defined by
\[
y = g(x) = \arcsin \left( (1 - x^2)^{-1/2} \right).
\]

Proceeding with the assumption that the function has a derivative we use the chain rule as follows. Let \(g = G \circ h\) and
\[
y = g(x) = G(h(x)) = G(w) = \arcsin(w) \quad \text{where } w = h(x) = (1 - x^2)^{-1/2}.
\]

Then
\[
g'(x) = G'(w)h'(x) \quad \text{where } w = h(x) = (1 - x^2)^{-1/2}
\]
\[
= \frac{1}{(1 - w^2)^{1/2}}h'(x) \quad \text{where } w = h(x) = (1 - x^2)^{-1/2}
\]
\[
= \frac{1}{(1 - (1 - x^2)^{-1})} \left[ x(1 - x^2)^{-3/2} \right]
\]
\[
= \frac{(1 - x^2)^{1/2}}{(1 - x^2)^{3/2}} \left[ x(1 - x^2)^{-3/2} \right]
\]

There is clearly a problem here, taking the square root of a negative number. Why? We assumed the function was defined as the above composition of two functions. Now the arcsin function is only defined for argument \(|w| \leq 1\). But \(w = (1 - x^2)^{-1/2} \leq 1\) only for \(x = 0\). This means the given function \(f\) is only defined on a domain of a single point, that is \(x = 0\). It therefore cannot be differentiated!

**QUESTION 6**

(a) We are given the following three functions defined on the interval \([\beta, \alpha]\):
\[
\begin{align*}
f_1(x) & = 2 \arcsin(\sqrt{|x - \beta|/|\alpha - \beta|}) \\
f_2(x) & = 2 \arctan(\sqrt{|x - \beta|/|\alpha - x|}) \\
f_3(x) & = \arcsin(2\sqrt{|\alpha - x|/|\alpha - \beta|}).
\end{align*}
\]
We shall calculate the derivative of \( f_3 \). Let \( f_3 = F \circ h \) where

\[
f_3(x) = (F \circ h)(x) = F(h(x)) = F(w) = \arcsin(w) \quad \text{where} \quad w = h(x) = \frac{2(\alpha - x)(x - \beta)^{1/2}}{[\alpha - \beta]}.
\]

This function is well defined as the composition on the given interval. Then by the chain rule the derivative of \( f_3 \) equals:

\[
\frac{df_3(x)}{dx} = F'(w)h'(x) \quad \text{where} \quad w = h(x)
\]

\[
= \frac{1}{1 - w^2} \left[ \frac{1}{\alpha - \beta} \left( -\frac{(x - \beta) - (\alpha - x) - (\alpha - x)(x - \beta) - (\alpha - x)^{1/2}}{((\alpha - \beta)^2 - 4(\alpha - x)(x - \beta))(x - \beta)^{1/2}} \right) \right]
\]

Now it is easy to verify that

\[
(\alpha - \beta)^2 - 4(\alpha - x)(x - \beta) = \alpha^2 + 2\alpha x + \beta^2 - 4(\alpha + \beta)x + 4x^2 = (\alpha + \beta - 2x)^2.
\]

Hence

\[
\frac{(\alpha - x)^{1/2}}{(x - \beta)^{1/2}} - \frac{(\alpha - x)^{1/2}}{(x - \beta)^{1/2}} = \frac{(\alpha - x)^{1/2} - (\alpha - x)^{1/2}}{(x - \beta)^{1/2}(\alpha - x)^{1/2}} = \frac{\alpha - \beta - 2x}{(x - \beta)^{1/2}(\alpha - x)^{1/2}}.
\]

Substituting for these results

\[
\frac{df_3(x)}{dx} = \frac{1}{((\alpha + \beta - 2x)^{1/2})(x - \beta)^{1/2}(\alpha - x)^{1/2}} = \frac{1}{((\alpha + \beta - 2x)^{1/2}(x - \beta)(\alpha - x)^{1/2})}.
\]

Similar calculations yields the same derivative of \( f_1 \) and \( f_2 \). For explanation consider a right angled triangle with sides \( AB = (x - \beta)^{1/2} \), \( AC = (\alpha - x)^{1/2} \) and hypotenuse \( BC = (\alpha - \beta)^{1/2} \). Let \( \theta \) be the angle between the sides \( AC \) and \( BC \). Then

\[
\sin(\theta) = \left( \frac{x - \beta}{\alpha - \beta} \right)^{1/2} \quad \tan(\theta) = \left( \frac{x - \beta}{\alpha - x} \right)^{1/2} \quad \text{and} \quad \cos(\theta) = \left( \frac{\alpha - x}{\alpha - \beta} \right)^{1/2}.
\]

Now

\[
\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \left( \frac{x - \beta}{\alpha - \beta} \right)^{1/2} \left( \frac{\alpha - x}{\alpha - \beta} \right)^{1/2} = \frac{2((\alpha - x)(x - \beta))^{1/2}}{\alpha - \beta}
\]

We now see that \( f_1(x) = f_2(x) = f_3(x) = 2\theta(x) \). The functions must all have the same derivatives.

(b) We are given two curves \( C_1 \) and \( C_2 \) described by \( 3y = 2x + x^4y^2 \) and \( 2y + 3x + y^5 = x^2y \), respectively. To find the tangents at the origin we use implicit differentiation.

For curve \( C_1 \) we determine differentiating implicitly

\[
3\frac{dy}{dx} = 2 + [4x^3]y^2 + x^4 \left[ 3y^2 \frac{dy}{dx} \right].
\]
Collecting terms:
\[(3 - 3x^4y^2) \frac{dy}{dx} = 2 + 4x^2y^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{2 + 4x^3y^3}{3 - 3x^4y^2}.\]

At the point \((0,0)\) the slope of the tangent \(m_1\) is simply \(2/3\).

For curve \(C_2\) we determine differentiating implicitly
\[2 \frac{dy}{dx} + 3 + 5y^4 \frac{dy}{dx} = [3x^2] y + x^3 \frac{dy}{dx}.\]

Collecting terms:
\[(2 - x^3 + 5y^4) \frac{dy}{dx} = -3 + 3x^2 y \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2y - 3}{2 - x^3 + 5y^4}.\]

At the point \((0,0)\) the slope of the tangent \(m_2\) is simply \(-3/2\).

Since \(m_1 m_2 = -1\), this shows that the two tangents to these curves at the origin at perpendicular to each other.

**(c)** We are required to show that
\[\ln(x) > 2x - x^2/2 - 3/2 \quad \text{whenever} \quad x > 1.\]

Consider the function \(f\) defined by
\[f(x) = \ln(x) - 2x - x^2/2 - 3/2 \quad \text{for} \quad x > 1.\]

Now simple differentiation yields
\[f'(x) = \frac{1}{x} - 2 + x = \left( \frac{1}{\sqrt{x}} - \sqrt{x} \right) 2.\]

So we find that \(f'(x) > 0\) for all \(x > 0\). Hence \(f\) is strictly increasing for \(x > 1\) so then \(f(x) > f(1)\) for all \(x > 1\). But \(f(1) = 0\). Hence
\[f(x) > 0 \quad \text{or} \quad \ln(x) > 2x - x^2/2 - 3/2 \quad \text{whenever} \quad x > 1.\]

**QUESTION 7**

The derivatives of the functions are given below. Differentiation details are not given. Please check your results.

(a) \(f'(x) = \arctan(x/a).\)
(b) \(f'(x) = x \arctan(x/a).\)
(c) \(f'(x) = (\arcsin(x/a))^2.\)
(d) \(f'(x) = x e^x [\ln(x^2) + (x \ln(x) + 1)].\)
(e) \(f'(x) = 1/(1 + e^x).\)
(f) \(f'(x) = e^x \sin(x).\)
(g) \(f'(x) = \ln \left( x + (x^2 - a^2)^{1/2} \right) / x^2.\)
QUESTION 8

(a) We are given the volume $V$ of water in the bowl when at a depth $x$ cm is

$$ V(x) = \pi x^2 (6 - x)/3 \text{ cm}^3. $$

The volume is decreasing at the constant rate of $3 \text{ cm}^3/\text{sec}$. Knowing that $x$ is itself a function of time, say $x = f(t)$, we can write the volume as composition of functions, indicating the volume as a function $W$ of time:

$$ V = W(t) = V(f(t)) = (V \circ f)(t). $$

Using the chain rule

$$ W'(t) = V'(x)f'(t) \text{ where } x = f'(t). $$

Now we are given $W'(t) = -3$, and calculate $V'(x)$ as $V'(x) = \pi x(4 - x)$. Hence

$$ -3 = \pi x(4 - x)f'(t). $$

At depth of 2 cm we solve this equation to find

$$ f'(t) = -3/(4\pi) \approx -0.23873 \text{ cm/sec}. $$

(b) The volume $V$ of the cylindrical tank is given by

$$ V = \pi r^2 h, $$

where both radius $r = R(t)$ and height $h = H(t)$, are functions of time. Using the product rule and chain rule,

$$ \frac{dV}{dt} = 2\pi rh \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}. $$

Now given radius $r = 2$ m and volume $V = 20\pi$ m$^3$, then $h = 5$ m,

The volume is increasing at the rate $dV/dt = 4\pi/5$ m$^3$/min and radius is increasing at the rate $dr/dt = 0.2/100 = 1/500$ m/min. Hence we find

$$ \frac{4\pi}{5} = 2\pi rh/500 + \pi r^2 \frac{dh}{dt} = \pi/25 + 4\pi \frac{dh}{dt}, $$

from which we determine the rate at which the height is increasing

$$ \frac{dh}{dt} = 0.19 \text{ m/min}. $$