CHAPTER 2
Basic Linear Algebra (MATH11163/11218)

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Contents

2 Vectors .......................... 1
  2.1 Basic Definitions .......................... 1
  2.2 Further properties and concepts .................. 3
  2.3 Vector decomposition and linear independence .............. 9
    2.3.1 Vector decomposition in the plane ................. 9
    2.3.2 Vector decomposition in Euclidean space ............ 10
    2.3.3 Linear dependence and linear independence .......... 11
    2.3.4 Types of vectors ................................ 16
  2.4 Rectangular cartesian coordinates .................. 17
    2.4.1 Parametrised equation of a plane ................. 24
    2.4.2 Cartesian equation of a line in $R^3$ .......... 27
  2.5 The Dot (Scalar or Inner) Product .................. 29
    2.5.1 Properties of the dot product .................... 30
    2.5.2 Coordinate representation of the dot product ....... 32
    2.5.3 Cartesian equation of a plane .................... 37
  2.6 The vector (cross) product ........................ 42
    2.6.1 Properties of the vector product ................. 43
    2.6.2 Cartesian form of the vector product .......... 45
    2.6.3 Vector area of a parallelogram .................. 48
2.6.4 Proof of the left distributive law

2.7 Products of more than two vectors

2.7.1 The scalar triple product

2.7.2 The vector triple product

2.8 Further exercises
Chapter 2

Vectors

2.1 Basic Definitions

Although our notation of what constitutes a vector will be considerably broadened in later courses to that of an abstract linear space, for the present we define a vector as follows:

DEFINITION 2.1
A vector $\mathbf{a}$ is a quantity which possesses magnitude and a direction property, and which adds according to a triangle law (see below).

Our notion of a vector is a geometrical one, a physical quantity in space which we take here as three dimensional Euclidean space, which we shall represent with the notation $\mathbb{R}^3$ or $\mathbb{E}$.

Graphically the vector may be represented by a line segment, the length of which is adjusted to some scale to indicate the desired magnitude. It has direction specified by the arrow together with a head and a tail as depicted in Figure 2.1.

Physically it may for example, denote a force, the position with respect to some reference point of some particle in space, or a particle’s velocity.

In the Figure vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are identical vectors having the same magnitude and the same direction or sense, even though they have a different tail point and head point. The direction or sense of the vector is indicated by the arrow on the straight line segment $AB$. This straight line is called the line of action of the vector. Sometimes in texts you will find other alternative notation used, such as $\vec{a}$, $\vec{AB}$. 
CHAPTER 2. VECTORS

Figure 2.1: Free vector - a directed line segment

Note that, with this definition, the position of the vector in space is immaterial. Such a vector is called a free vector.

The Triangle Law

Given two vectors $\vec{a}$ and $\vec{b}$ the sum of the two vectors $\vec{c}$ written $\vec{c} = \vec{a} + \vec{b}$ is defined in magnitude and direction, by the following triangle construction in Figure 2.2.

Figure 2.2: The Triangle Law of vector addition

We note that the order of addition is immaterial thus indicating that the operation of taking the sum of two vectors is commutative, that is,

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}.$$
2.2 Further properties and concepts

In this section some basic properties of vectors are developed along with some other notational concepts. First we define:

DEFINITION 2.2 Scalar

Quantities which have magnitude only are called scalars.

Examples of scalars are mass, distance and volume. A scalar can be represented by a number with an associated sign, which indicates the magnitude according to some convenient scale.

Now to some simple properties.

1. **Magnitude of a vector.** The magnitude of \( \vec{a} \) denoted by \( |\vec{a}| \) is the length of the corresponding line segment \( AB \).

2. **Equality of vectors.** Two vectors \( \vec{a} \) and \( \vec{b} \) are said to be equal if they are equal in both magnitude and direction, and we write \( \vec{a} = \vec{b} \).

3. **The null vector.** The null vector \( \vec{0} \) is that vector which has zero magnitude and therefore undefined direction. It is represented by a point.

4. **Multiplication of vectors by scalars.** Given scalar \( \lambda \) together with vector \( \vec{a} \), the product of a scalar and a vector \( \lambda \vec{a} \), is defined to be that vector \( \vec{c} = \lambda \vec{a} \) which has:
   
   (i) Magnitude equal to the magnitude of \( \lambda \) multiplied by the magnitude of \( \vec{a} \), that is,
   
   \[ |\vec{c}| = |\lambda||\vec{a}|, \quad \text{and} \]
   
   (ii) Direction specified by the same line of action as \( \vec{a} \) and same sense if \( \lambda > 0 \), or opposite sense if \( \lambda < 0 \). See Figure 2.3.

   Hence we define the negative vector \( -\vec{a} = (-1)\vec{a} \). It has the same length as \( \vec{a} \) but the opposite sense.

5. **Subtraction of vectors.** The difference of two vectors \( \vec{a} \) and \( \vec{b} \), namely \( \vec{a} - \vec{b} \) can now be defined in terms of addition:

   \[ \vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}. \]
The following properties (and laws) can be shown true by simple geometrical arguments from the given physical definition of a vector. They are set as an exercise. Let $V$ be the set of all vectors $\mathbf{a}$ as defined above. We say that $\mathbf{a}$ is an element of $V$ or belongs to $V$, and write $\mathbf{a} \in V$.

**EXERCISE 2.1**
Using simple geometrical arguments establish the following properties and laws for vectors.

(A) **Closure** of the operation of addition ‘$+$’. Given any two vectors $\mathbf{a}$ and $\mathbf{b}$ in $V$, then $\mathbf{c} = \mathbf{a} + \mathbf{b} \in V$.

(B) **Commutative Law of Addition**. Given any $\mathbf{a} \in V$ and $\mathbf{b} \in V$, then

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$  
(This law indicates that the sum of two vectors is independent of the order in which the two vectors are added.)

(C) **Associative Law of Addition**. Given any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$, then

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$  
(This law indicates that the sum of three vectors is independent of the order in which the vectors are added. We may remove the bracketing in such addition.)

(D) **Additive Identity**. There exists a vector $\mathbf{0}$, called the identity vector for addition, or the null vector, such that

$$\mathbf{a} + \mathbf{0} = \mathbf{a} = \mathbf{0} + \mathbf{a} \text{ for all } \mathbf{a} \in V.$$
2.2. FURTHER PROPERTIES AND CONCEPTS

(E) Additive inverse. To each vector \( \vec{a} \in V \), there exists a vector called the additive inverse, written \(-\vec{a}\), such that
\[
\vec{a} + (-\vec{a}) = \emptyset = (-\vec{a}) + \vec{a}.
\]
It should be clear that \(-\vec{a} = (-1)\vec{a}\).

(F) Closure under scalar multiplication. Given any scalar \( \lambda \) and any vector \( \vec{a} \in V \) then \( \lambda \vec{a} \in V \).

(G) Given scalars \( \lambda \) and \( \mu \) then
\[
(\lambda + \mu)\vec{a} = \lambda \vec{a} + \mu \vec{a} \text{ for all } \vec{a} \in V.
\]

(H) Given scalars \( \lambda \) and \( \mu \) then
\[
\lambda(\mu\vec{a}) = \mu(\lambda\vec{a}) = (\lambda \mu)\vec{a} \text{ for all } \vec{a} \in V.
\]

(I) Given the multiplicative scalar identity 1, then
\[
1\vec{a} = \vec{a} \text{ for all } \vec{a} \in V.
\]

(J) Distributive law of scalar multiplication over vector addition. Given scalar \( \lambda \) then
\[
\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} \text{ for all } \vec{a}, \vec{b} \in V.
\]

A set \( V \) of elements which obey the axioms (A)-(J) is said to be a linear vector space. This important vector space will be studied in more depth for those who continue on with their studies in matrix algebra.

The following definition defines the concept of position vector and how to define position vectors in making a change of origin.

**DEFINITION 2.3 Position Vector and change of origin**

Given a point in space \( \vec{O} \) which we take as the origin, see Figure 2.4, we define the vector \( \vec{a} = \vec{OA} \) to be the position vector of \( A \) relative to \( \vec{O} \). Given the point \( \vec{B} \) with position vector \( \vec{b} \), we say the vector \( \vec{BA} = \vec{b} - \vec{a} \) as given by the Triangle Law, defines the position vector of \( B \) relative to \( \vec{A} \).

**DEFINITION 2.4 Unit vector**

A vector \( \vec{a} \) is said to be a unit vector if it has unit magnitude, that is, \( |\vec{a}| = 1 \). The vector \( \vec{a}/|\vec{a}| \) is a unit vector with direction specified by the vector \( \vec{a} \).
DEFINITION 2.5 Parallel vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if they have the same direction in space.

This means that parallel vectors do not necessarily have the same magnitude, but the definition implies that if two vectors $\vec{a}$ and $\vec{b}$ are parallel, then there exists a scalar $\lambda \neq 0$ such that $\vec{a} = \lambda \vec{b}$. If $\lambda > 0$ then the vectors are said to be parallel with the same sense, if $\lambda < 0$ then the vectors are said to be parallel with the opposite sense. Vectors $\vec{a}$ and $2\vec{a}$ are parallel vectors with the same sense. Hence the magnitudes of the two vectors are connected by the equation $|\vec{a}| = |\lambda||\vec{b}|$ for some scalar $\lambda$.

DEFINITION 2.6 Orthogonal vectors

Two vectors $\vec{a}$ and $\vec{b}$ are said to be orthogonal (or perpendicular) if the angle between their lines of action is $90^\circ$ (or $\pi/2$). As the two vectors are free vectors this angle is defined when their tails are placed at the same point. See Figure 2.5.
2.2. FURTHER PROPERTIES AND CONCEPTS

We begin our study with a discussion on how to represent the simplest of structures, namely the straight line. How we describe the locus of points which constitute a straight line parametrically is shown in the next example.

EXAMPLE 2.1
Determine the parametric equation of the straight line which passes through point \( A \) and has direction parallel to the vector \( \vec{b} \). See Figure 2.6.

Solution. Let \( P \) be a typical point on this line through \( A \) and let \( \vec{OA} = \vec{a} \), and \( \vec{OP} = \vec{r} \) relative to the Origin \( O \). Then for some value of the scalar \( t \), by the triangle law of vector addition,

\[
\vec{OP} = \vec{r}(t) = \vec{a} + t\vec{b},
\]

since \( \vec{AP} \) is parallel to \( \vec{b} \).

Indeed for each point \( P \) there must exist some scalar/number \( t \) such that this is so, and vice versa, for each \( t \) there does exist a point \( P \) on the line defined by the formula. Therefore to make this correspondence clear we write

\[
\vec{r}(t) = \vec{a} + t\vec{b}, \tag{2.1}
\]

where \( t \) is a parameter taking real values, that is, \( t \in R \), where \( R \) represent the set of real numbers.
The locus of all points obtained with position vectors \( r(t) \) relative to the Origin \( O \) as \( t \) varies through all real values is the straight line passing through \( A \) parallel to the vector \( \vec{b} \). This vector equation 2.1 is said to define parametrically the straight line with parameter \( t \), and is called the parametric equation of the straight line. We call \( r \) a vector function of the parameter \( t \), and write its value as \( \vec{r}(t) \) at each given value of \( t \).

\[ \vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a}), \quad \text{with} \quad t \in \mathbb{R}, \]

\[ \vec{r}(t) = (1 - t)\vec{a} + t\vec{b}, \quad \text{with} \quad t \in \mathbb{R}. \] (2.2)

\[ \square \]

**NOTE**: A straight line CANNOT be represented by just a simple vector. It must be represented by a vector function.

**EXAMPLE 2.2**

*Determine the parametric equation of the straight line which passes through points \( A \) and \( B \) with positions vectors \( \vec{a} \) and \( \vec{b} \) relative to origin \( O \). See Figure 2.7.*

**Solution.** Considering Example 2.1, we need to find a vector to which the line is parallel. Clearly this vector can be taken to be

\[ \vec{AB} = \vec{b} - \vec{a}, \quad \text{or} \quad \vec{BA} = \vec{a} - \vec{b}. \]

Therefore the parametric equation of the line can be written:

\[ \vec{r}(t) = \vec{a} + t\vec{AB} = \vec{a} + t(\vec{b} - \vec{a}), \quad \text{with} \quad t \in \mathbb{R}, \]

\[ \vec{r}(t) = (1 - t)\vec{a} + t\vec{b}, \quad \text{with} \quad t \in \mathbb{R}. \] (2.2)
Here \( t \) is a real parameter. Observing the comment above we could have also written the parametric equation as

\[
\mathbf{r}_2(s) = \mathbf{a} + s \mathbf{BA} = \mathbf{a} + s(\mathbf{a} - \mathbf{b}), \quad \text{with } s \in \mathbb{R},
\]

\[
(1 + s)\mathbf{a} - sb, \quad \text{with } s \in \mathbb{R}.
\] (2.3)

As the parameter \( s \) takes all real values the same locus of points will be obtained plotting \( \mathbf{r}_2(s) \) as will be obtained using the former representation Equation 2.2 using the parameter \( t \). Observe that

\[
\mathbf{r}_1(0) = \mathbf{a} = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1(1) = \mathbf{b} = \mathbf{r}_2(-1),
\]

and

\[
\mathbf{r}_1(t) = \mathbf{a} + t \mathbf{AB} \quad \text{with } t \in [0, 1],
\]

describes all points on the line segment \( AB \).

\[\square\]

### 2.3 Vector decomposition and linear independence

#### 2.3.1 Vector decomposition in the plane

We begin with decomposition in the plane. Consider any two non-zero and non-parallel vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^2 \). Then for any two scalars \( \lambda \) and \( \mu \) form the vector

\[
\mathbf{r}(\lambda, \mu) = \lambda \mathbf{a} + \mu \mathbf{b}.
\]

The vector \( \mathbf{r} \) is said to be a **linear combination** of vectors \( \mathbf{a} \) and \( \mathbf{b} \). See Figure 2.8. The set of all points defined with position vector \( \mathbf{r} \)

\[
\{ \mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} : \lambda, \mu \in \mathbb{R} \},
\]

is called the **plane spanned** by the non-zero, non-parallel vectors \( \mathbf{a} \) and \( \mathbf{b} \).

Any point \( C \) in the plane of \( \mathbf{a} \) and \( \mathbf{b} \) can be expressed in the form above for we can construct a parallelogram \( OACB \) with \( \mathbf{c} \) as its diagonal and its edges parallel to \( \mathbf{a} \) and \( \mathbf{b} \).

We say that vector \( \mathbf{c} \) is **linearly dependent** on vectors \( \mathbf{a} \) and \( \mathbf{b} \), or that the set of vectors \( \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} \) is linearly dependent. The two vectors \( \mathbf{a} \) and \( \mathbf{b} \) are said to be
linearly independent. In this case it is not possible to write \( \mathbf{a} \) as a multiple of \( \mathbf{b} \). The plane is said to be two dimensional as it requires a maximum of two such non-zero, non-parallel vectors, that is, two independent vectors to describe it. We might think of it otherwise that it requires two independent parameters \( \lambda \) and \( \mu \) to describe all the points lying on the plane spanned by the vectors \( \mathbf{a} \) and \( \mathbf{b} \). (The concept of dimension will be more clearly defined in a later course in Mathematics.)

Note that the plane of points formed in such a way from the vectors \( \mathbf{a} \) and \( \mathbf{b} \) is unique but there are many vectors which can span this plane. For example, this plane is also spanned by the vectors \( \mathbf{a} - \mathbf{b} \) and \( \mathbf{a} + \mathbf{b} \). Verify this by finding scalars \( \beta \) and \( \eta \) such that the position vector to any point \( \mathbf{c} \) on the plane can be written

\[
\mathbf{c} = \beta (\mathbf{a} - \mathbf{b}) + \eta (\mathbf{a} + \mathbf{b}).
\]

### 2.3.2 Vector decomposition in Euclidean space

Suppose we are given any three non-zero vectors \( \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \). Let us assume that not all vectors are parallel to a single plane. Then if \( \mathbf{d} \in \mathbb{R}^3 \) is any other vector, \( \mathbf{d} \) can be expressed as a linear combination or linear function of the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \). That is, there exist scalars \( \lambda, \mu \) and \( \eta \), such that

\[
\mathbf{d}(\lambda, \mu, \eta) = \lambda \mathbf{a} + \mu \mathbf{b} + \eta \mathbf{c}.
\]

The construction by geometry to verify this is as follows.

Any point \( P \) in \( \mathbb{R}^3 \) can be expressed in the form above for we can construct a parallelopiped \( OACBD \), see Figure 2.9, with \( \mathbf{d} \) as its diagonal and its edges parallel to \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \).
2.3. VECTOR DECOMPOSITION AND LINEAR INDEPENDENCE

We say that vector $\mathbf{d}$ is linearly dependent on vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or that the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ is linearly dependent. The three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are said to be linearly independent.

The vector space $\mathbb{R}^3$ is said to be spanned by the three given vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The space is said to have dimension three (3). Clearly such a space cannot be spanned by just two vectors that are non-zero and not parallel. Further it should be clear that we can choose another set of three vectors in a like manner to span $\mathbb{R}^3$; so the choice of independent vectors is not unique. However:

**REMARK 2.1**

The vector decomposition described above is UNIQUE; that is, the ordered tuplets $(\lambda, \mu)$ and $(\lambda, \mu, \eta)$ are unique for the given vectors in each case, the case of the plane and $\mathbb{R}^3$.

2.3.3 Linear dependence and linear independence

In the case of the plane defined above we have described the two vectors $\mathbf{a}$ and $\mathbf{b}$ as linearly independent. Similarly for $\mathbb{R}^3$, the three described vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are said to be linearly independent.
The precise definition of linear dependence and linear independence, which can be extended to higher dimensional vector spaces is as follows:

**DEFINITION 2.7**

A set of three non-zero vectors \( \vec{a}, \vec{b}, \vec{c} \) is said to be a linearly dependent set if there exists scalars \( \lambda_1, \lambda_2, \lambda_3 \) not all zero, and

\[
\lambda_1 \vec{a} + \lambda_2 \vec{b} + \lambda_3 \vec{c} = \vec{0}.
\]  
(2.4)

If the set is not linearly dependent it is said to be linearly independent. In this case all \( \lambda_k = 0, k = 1, 2, 3 \), and no linear combination such as Equation 2.4 exists.

This definition has been stated for three vectors but can be easily adjusted for two vectors and four vectors, and for that matter any finite number of vectors.

With respect to the above cases we present discussions on two cases:

**Plane discussion**

Recall from Section 2.3.1 that vector \( \vec{a} \) is not parallel to \( \vec{b} \). We see that \( \vec{c} = \lambda \vec{a} + \mu \vec{b} \) can be written as

\[
(1) \vec{c} + (-\lambda) \vec{a} + (-\mu) \vec{b} = \vec{0}.
\]

If we define \( \lambda_1 = 1, \lambda_2 = -\lambda \) and \( \lambda_3 = -\mu \), then not all \( \lambda_k \) are zero since one of them is definitely equal to 1. The definition of linear dependence is satisfied, so the set \( \{ \vec{a}, \vec{b}, \vec{c} \} \) is linearly dependent.

Now what about vectors \( \vec{a} \) and \( \vec{b} \)? Suppose there were two scalars \( \lambda_1 \) and \( \lambda_2 \) not both zero such that

\[
\lambda_1 \vec{a} + \lambda_2 \vec{b} = \vec{0}.
\]

One of these two scalars \( \lambda_1 \) and \( \lambda_2 \) is non-zero by definition. Suppose it is \( \lambda_1 \). Then

\[
\vec{a} = -\frac{\lambda_2}{\lambda_1} \vec{b}.
\]

This implies that the vector \( \vec{a} \) is parallel to \( \vec{b} \) which then contradicts the assumption that \( \vec{a} \) and \( \vec{b} \) are non-parallel vectors. Hence by contradiction argument, the only way that \( \lambda_1 \vec{a} + \lambda_2 \vec{b} = \vec{0} \) can be satisfied, is for \( \lambda_1 = 0 = \lambda_2 \). We conclude by definition that \( \vec{a} \) and \( \vec{b} \) from a linearly independent set.
2.3. VECTOR DECOMPOSITION AND LINEAR INDEPENDENCE

$R^3$ discussion

Recall that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not all parallel to the same plane. It is easy to see that for vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$,

$$(1)\vec{d} + (-\lambda)\vec{a} + (-\mu)\vec{b} + (-\eta)\vec{c} = \vec{0}.$$ 

So taking values $\lambda_1 = 1 \neq 0$, $\lambda_2 = -\lambda$, $\lambda_3 = -\mu$ and $\lambda_4 = -\eta$, the definition is satisfied to show these four vectors form a linearly dependent set.

We show that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in this case form a linearly independent set by contradiction argument. Suppose there exists $\lambda_1$, $\lambda_2$ and $\lambda_3$ not all zero such that

$$\lambda_1\vec{a} + \lambda_2\vec{b} + \lambda_3\vec{c} = \vec{0}.$$ 

Without loss of generality suppose $\lambda_1 \neq 0$. Then

$$\vec{a} = -\lambda_2\vec{b} + \lambda_3\vec{c}/\lambda_1$$

$$= -\left(\frac{\lambda_2}{\lambda_1}\right)\vec{b} - \left(\frac{\lambda_3}{\lambda_1}\right)\vec{c}.$$ 

This equation tells us that the vector $\vec{a}$ lies in the plane spanned by $\vec{b}$ and $\vec{c}$. So this contradicts the assumption that the three vectors are not all parallel to one plane. We of course assume here that $\vec{b}$ is not parallel to $\vec{c}$. For if this was the case then $\vec{c}$ would be parallel to the plane spanned by the vectors $\vec{a}$ and $\vec{b}$ which again contradicts the assumption. No two vectors of this set can be parallel. The set $\{\vec{a}, \vec{b}, \vec{c}\}$ is therefore linearly independent as $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Let us now see how we can use these vector concepts in proving some known properties in geometry.

EXAMPLE 2.3

Suppose we are given two points $A$ and $B$ with position vectors $\vec{a}$ and $\vec{b}$ relative to the origin $O$, see Figure 2.10. What is the position vector $\vec{r}$ of the point $P$ which divides the line segment $AB$ internally in the ratio of $m : n$.

Solution. If the point $P$ divides the line segment $AB$ in the ratio $m : n$, this means that

$$AP : PB = m : n \text{ or } \frac{AP}{PB} = \frac{m}{n}.$$ 

Further it means that

$$\frac{AP}{AB} = \frac{m}{m + n}.$$
Given origin \( O \), the position vector \( \mathbf{r} \) of point \( P \) can be determined by the triangle law of addition as follows:

\[
\mathbf{r} = \mathbf{OA} + \mathbf{AP} = \mathbf{a} + \frac{\mathbf{AP}}{AB} \mathbf{AB} = \mathbf{a} + \frac{m}{m+n} (\mathbf{b} - \mathbf{a}) = \frac{(m+n)\mathbf{a} + m(\mathbf{b} - \mathbf{a})}{m+n} = \frac{na + mb}{m+n}.
\]  

(2.5)

Figure 2.10: Division of line segment \( AB \) in ratio \( m : n \)

From this last example we can now say that the midpoint of \( AB \) has position vector

\[
\mathbf{r} = (\mathbf{a} + \mathbf{b})/2,
\]

since we take \( m = n = 1 \).

The point \( P \) which divides \( AB \) in the ratio \( 1 : 2 \) is

\[
\mathbf{r} = (2\mathbf{a} + \mathbf{b})/3,
\]

since we take \( m = 1, n = 2 \).

**EXAMPLE 2.4**

*Show by vector methods that the diagonals of a parallelogram bisect each other.*
2.3. VECTOR DECOMPOSITION AND LINEAR INDEPENDENCE

Solution. Select our origin at the corner of the parallelogram as shown in Figure 2.11. Let vectors \( \vec{a} \) and \( \vec{c} \) define the parallelogram as shown. If the origin is elsewhere we can just make a change of origin in the following argument. Our first objective is to find the point of intersection of the diagonal lines \( \ell_1 \) and \( \ell_2 \).

First we define the parametric equations of lines \( \ell_1 \) and \( \ell_2 \).

Line \( \ell_1 \):

\[
\vec{r}_1(t) = \vec{0} + t\vec{OB} = \vec{0} + t(\vec{a} + \vec{c}) \quad \text{with} \quad -\infty < t < \infty.
\]

Line \( \ell_2 \):

\[
\vec{r}_2(s) = \vec{a} + s\vec{AC} = \vec{a} + s(\vec{c} - \vec{a}) \quad \text{with} \quad -\infty < s < \infty.
\]

Now at the point of intersection there must exist values of the scalar parameters \( t \) and \( s \) such that

\[
\vec{r}_1(t) = \vec{r}_2(s).
\]

This implies that

\[
t(\vec{a} + \vec{c}) = \vec{a} + s(\vec{c} - \vec{a}), \quad \text{or}
\]

\[
(1 - s - t)\vec{a} + (s - t)\vec{c} = \vec{0}.
\]
The two vectors \( \mathbf{a} \) and \( \mathbf{c} \) must be linearly independent, otherwise the parallelogram would not be well defined. Hence by the definition of linear independence the coefficients in this expansion must equal zero, that is

\[
1 - s - t = 0 \quad \text{and} \quad s - t = 0.
\]

Solving we find that \( s = t = 1/2 \). Hence the point of intersection \( I \) has position vector

\[
\mathbf{r}_I(1/2) = (1/2)(\mathbf{a} + \mathbf{c}) = \mathbf{a} + (1/2)(\mathbf{c} - \mathbf{a}).
\]

This shows us that \( I \) defines the point of bisection of the two diagonal lines.

\[\Box\]

### 2.3.4 Types of vectors

We have now introduced the basic concept of a vector. There are many types of vectors that are commonly referred to. Here is some terminology you may not be familiar with from previous study.

1. A **free** vector can act anywhere in space; it is only necessary to preserve its magnitude and direction.

2. A **fixed** vector is one that acts at a particular point in space.

3. **Coplanar** vectors all lie in the same plane.

4. **Collinear** vectors have the same direction and the same line of action.

5. **Concurrent** vectors have lines of action that pass through the same point.

6. **Equal** vectors are those that are equal in magnitude and direction.

**Force as a vector**

When an external load is applied to the surface of a body, it is actually distributed over a finite area. In cases where this area is small compared to the size of the body, it may become difficult to measure the exact area of contact or to determine the loading distribution. As a result, computations regarding the effect of the load can be greatly simplified if the load is represented as a single concentrated force which acts only at a point on the body. Experimentally it has been shown that a concentrated force is a vector quantity, since it adds according to the parallelogram
or triangle law, and when it is applied to the body, it has a specific magnitude, direction and point of application. In the standard international system of units (SI) the magnitude of force is expressed in Newtons, in the British FPS system, it is expressed in pounds.

**EXERCISE 2.2**

Suppose we are given a triangle $ABC$ formed by three position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ relative to a given fixed origin $O$. There are three medians for this triangle. The median at $A$ is defined as that line segment passing through $A$ to the midpoint of the opposite side of the triangle $BC$. Medians at $B$ and $C$ are similarly defined. Show that the medians of the triangle have a common point of intersection using vector methods.

### 2.4 Rectangular cartesian coordinates

In this section we further develop the basic concepts of vectors into component form. It should be clear that special sets of vectors allow us to describe in a general way any vector in a plane and in Euclidean space. We call these vectors *basis* vectors.

![Mutually orthogonal unit basis vectors for $\mathbb{R}^3$](figure2.12.png)

**Figure 2.12: Mutually orthogonal unit basis vectors for $\mathbb{R}^3$**

**DEFINITION 2.8**

A set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ in $E = \mathbb{R}^3$ is called a basis for $\mathbb{R}^3$ if

(i) The set is linearly independent, and

(ii) The set of vectors is a spanning set for $\mathbb{R}^3$, that is, any vector $\mathbf{z} \in \mathbb{R}^3$ can be expressed as linear combination of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. 

We have seen previously that such a basis does in fact exist. The choice is ours to make further mathematics as simple as possible. It is for this reason we choose a set of mutually orthogonal, unit basis vectors \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} to take advantage of some simple Pythagorean geometry. By mutually orthogonal we imply the three basis vectors are perpendicular to each other, as depicted in Figure 2.12. The vectors are unit vectors so \(|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1\).

Any vector may be represented uniquely by the ordered number triplet \((a_1, a_2, a_3)\) as the decomposition is unique. Here \(a_1\mathbf{i}\) is the component of \(\mathbf{a}\) in the direction of \(\mathbf{i}\); it has length \(a_1\). We further observe from triangle OPA, see Figure 2.13, that

\[ a_1 = |a| \cos(\theta_1), \]

where \(\theta_1\) is the angle between the two vectors \(a\) and \(\mathbf{i}\). A similar discussion can be made for the other two components \(a_2\mathbf{j}\) and \(a_3\mathbf{k}\) with angles \(\theta_2\) and \(\theta_3\) appropriately defined. The scalars \(a_1\), \(a_2\) and \(a_3\) are called the Cartesian coordinates of the vector \(\mathbf{a}\) with respect to the basis set \(\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}\).

Figure 2.13: Coordinates of point \(A\) for the basis

We can therefore write using the triangle law of addition that

\begin{align*}
\mathbf{a} &= |a| \cos(\theta_1)\mathbf{i} + |a| \cos(\theta_2)\mathbf{j} + |a| \cos(\theta_3)\mathbf{k} \\
&= |a| \left(\gamma_1\mathbf{i} + \gamma_2\mathbf{j} + \gamma_3\mathbf{k}\right) \\
&= |\mathbf{a}| \left(\gamma_1\mathbf{i} + \gamma_2\mathbf{j} + \gamma_3\mathbf{k}\right) \quad (2.6)
\end{align*}
where
\[
\gamma_1 = \cos(\theta_1) = \frac{a_1}{\|a\|}, \\
\gamma_2 = \cos(\theta_2) = \frac{a_2}{\|a\|}, \\
\gamma_3 = \cos(\theta_3) = \frac{a_3}{\|a\|}.
\]
The coefficients \(\gamma_k, k = 1, 2, 3\) are called the direction cosines of \(\vec{a}\), and they specify uniquely the direction of \(\vec{a}\). Note that \(\gamma_1 \vec{i} + \gamma_2 \vec{j} + \gamma_3 \vec{k}\) is a unit vector and by Pythagoras’ theorem, see Figure 2.13.

\[
\|\vec{a}\| = \left(\hat{a}_1^2 + \hat{a}_2^2 + \hat{a}_3^2\right)^{1/2}.
\]

**REMARK 2.2**

The development of the Cartesian coordinate system may or may not be known to you. It is important that you realise the \(\{\vec{i}, \vec{j}, \vec{k}\}\) set is special as it forms a basis in three dimensional Euclidean space. Also the unit vectors are unit vectors and are mutually orthogonal. They are said to form an orthonormal set. Clearly there are many bases that we could select, indeed any set of three non-zero vectors not all parallel to a given plane will do.

The advantage of choosing the set \(\{\vec{i}, \vec{j}, \vec{k}\}\), is purely because of its geometric advantages (like using Pythagoras’ theorem) that come with it, and of course the easy calculations of the coordinates (and components) of a given vector.

With this new representation of a vector we can easily obtain its magnitude and direction. How does it affect addition of vectors and multiplication of a vector by a scalar? Suppose we are given two vectors, namely:

\[
\vec{x} = x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} \quad \text{and} \quad \vec{y} = y_1 \vec{i} + y_2 \vec{j} + y_3 \vec{k}.
\]

What is the coordinate representation of the sum of these two vectors \(\vec{u} = \vec{x} + \vec{y}\)? Using the basic laws obtained from the geometric definition the addition is obtained as in the following calculations:

\[
\vec{u} = (x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k}) + (y_1 \vec{i} + y_2 \vec{j} + y_3 \vec{k}) \\
= (x_1 \vec{i} + ((x_2 \vec{j} + x_3 \vec{k}) + (y_1 \vec{i} + (y_2 \vec{j} + y_3 \vec{k}))) \quad \text{Associative law} \\
= x_1 \vec{i} + (((x_2 \vec{j} + x_3 \vec{k}) + y_1 \vec{i}) + (y_2 \vec{j} + y_3 \vec{k})) \quad \text{Associative law} \\
= x_1 \vec{i} + ((y_1 \vec{i} + (x_2 \vec{j} + x_3 \vec{k}) + (y_2 \vec{j} + y_3 \vec{k})) \quad \text{Commutative law} \\
= x_1 \vec{i} + (y_1 \vec{i} + ((x_2 \vec{j} + x_3 \vec{k}) + (y_2 \vec{j} + y_3 \vec{k})) \quad \text{Commutative law} \\
= x_1 \vec{i} + (y_1 \vec{i} + (x_2 \vec{j} + x_3 \vec{k}) + (y_2 \vec{j} + y_3 \vec{k})) \quad \text{Associative law}
\]
CHAPTER 2. VECTORS

\[ (x_1 \hat{i} + y_1 \hat{j}) + ((x_2 \hat{j} + x_3 \hat{k}) + (y_2 \hat{j} + y_3 \hat{k})) \]  
Associative law

\[ (x_1 + y_1) \hat{i} + ((x_2 \hat{j} + x_3 \hat{k}) + (y_2 \hat{j} + y_3 \hat{k})) \]  
Law (G)

\[ (x_1 + y_1) \hat{i} + ((x_2 \hat{j} + x_3 \hat{k}) + (y_2 \hat{j} + y_3 \hat{k})) \]  
Associative Law

\[ (x_1 + y_1) \hat{i} + ( (x_2 \hat{j} + x_3 \hat{k}) + y_3 \hat{k}) \]  
Commutative Law

\[ (x_1 + y_1) \hat{i} + ((x_2 \hat{j} + x_3 \hat{k}) + (y_3 \hat{k})) \]  
Associative Law

\[ (x_1 + y_1) \hat{i} + ( (x_2 \hat{j} + (x_2 \hat{j} + x_3 \hat{k}) + y_3 \hat{k}) \]  
Law (G)

\[ (x_1 + y_1) \hat{i} + ( (x_2 \hat{j} + x_3 \hat{k}) + y_3 \hat{k}) \]  
Removing the brackets

These calculations have been written out in explicit detail to show how the various laws and rules apply. The end result is straightforward.

To find the sum of two vectors written in Cartesian form, one simply adds the vectors component wise, resulting in a vector whose coordinates are simply the sums of the coordinates of the two vectors in the basis vector directions.

Let us now consider multiplication of a vector by a scalar \( \lambda \). Take the vector \( \mathbf{v} = \lambda \mathbf{x} \). What is its coordinate representation? The calculations proceed as follows.

\[ \mathbf{v} = \lambda (x_1 \hat{i} + x_2 \hat{j} + x_3 \hat{k}) \]

\[ = \lambda (x_1 \hat{i} + x_2 \hat{j}) + x_3 \hat{k} \]  
Associative Law

\[ = \lambda (x_1 \hat{i} + x_2 \hat{j}) + \lambda x_3 \hat{k} \]  
Distributive Law (J)

\[ = (\lambda x_1 \hat{i} + x_2 \hat{j} + (x_3 + y_3) \hat{k}) \]  
Distributive Law (J)

\[ = (\lambda x_1 \hat{i} + \lambda x_2 \hat{j} + \lambda x_3 \hat{k}) \]  
Removing brackets

Hence the cartesian representation of vector multiplied by a scalar is simply that vector which is obtained by multiplying each of the respective coordinates by the scalar.

Representation of a vector as a matrix in \( \mathcal{M}(3 \times 1) \)

It is appropriate to recall here properties of matrix addition and multiplication by a scalar. We introduced the notation of a vector in the previous chapter to be a column vector belonging to \( \mathcal{M}(n \times 1) \). Our study here is in three dimensional Euclidean space and \( n = 3 \).
Consider scalar \( \lambda \) and vectors \( \mathbf{x} \) and \( \mathbf{y} \) belonging to \( M(3 \times 1) \) with representation

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},
\]

where the elements in the arrays for \( \mathbf{x} \) and \( \mathbf{y} \) are the unique coordinates of the vectors represented with respect to the \( \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \} \) basis. Then using the properties of matrix addition and multiplication of matrices by scalars:

\[
\mathbf{u} = \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix},
\]

and

\[
\lambda \mathbf{x} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{bmatrix}.
\]

The new coordinates for \( \mathbf{u} = \mathbf{x} + \mathbf{y} \) and \( \lambda \mathbf{x} \) obtained using properties of matrices are consistent with that obtained in the discussion above using the representation of the vector as a linear combination of the basis vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \). Each basis vector has the following matrix representation

\[
\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

The vector \( \mathbf{x} \) can now be written as a linear combination of the basis vectors in matrix form

\[
\mathbf{x} = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

We will use the matrix representation when appropriate.

Let us consider a simple practical problem.

**Example 2.5**

Three forces \( \mathbf{F}_1 = 2 \mathbf{i} + 3 \mathbf{j} - \mathbf{k} \), \( \mathbf{F}_2 = -5 \mathbf{i} + \mathbf{j} + 2 \mathbf{k} \) and \( \mathbf{F}_3 = 6 \mathbf{i} - \mathbf{k} \) (Newtons) act simultaneously on a particle. The resultant force \( \mathbf{F} \) on the particle is the vector sum of the three forces, find the magnitude and direction of this resultant force.
Solution. The resultant force \( \vec{F} \) being the vector sum of the three forces can be calculated as:

\[
\vec{F} = F_1 + F_2 + F_3
\]

\[
= (2\hat{i} + 3\hat{j} - \hat{k}) + (-5\hat{i} + \hat{j} + 2\hat{k}) + (6\hat{i} - \hat{k})
\]

\[
= (2 - 5 + 6)\hat{i} + (3 + 1 + 0)\hat{j} + (-1 + 2 - 1)\hat{k}
\]

\[
= 3\hat{i} + 4\hat{j}.
\]

Using the above formula the magnitude of the force is given by

\[
|\vec{F}| = \sqrt{(3^2 + 4^2 + 0^2)} = 5 \text{ (Newtons)}.
\]

Since the \( z \) coordinate of the force \( \vec{F} \) is zero, the resultant force is parallel to the \( ij \)-plane. The direction of the force is in the first quadrant of the plane, making an angle \( \theta_1 \) with the \( \hat{i} \) axis, where \( \cos(\theta_1) = 3/5 \), so that \( \theta_1 \approx 53.13^\circ \). Similarly we calculate \( \cos(\theta_2) = 4/5 \) and \( \theta_2 \approx 36.87^\circ \) Hence the direction cosines are

\[
\gamma_1 = 3/5, \quad \gamma_2 = 4/5, \quad \gamma_3 = 0.
\]

We may write the resultant force \( \vec{F} \) in the form \( \vec{F} = |\vec{F}| \hat{n} \) where \( |\vec{F}| = 5 \text{ (Newtons)} \) is the force magnitude and \( \hat{n} = (3\hat{i} + 4\hat{j})/5 \) is a unit vector specifying the direction of the resultant force.

Our next example revisits some of the linear dependence and independence concepts of the previous section, but this time with a knowledge of a coordinate system to express uniquely each of the vectors.

**EXAMPLE 2.6**

Given vectors \( \vec{a} = 2\hat{i} + \hat{j}, \vec{b} = 3\hat{i} - \hat{j} + \hat{k}, \vec{c} = -\hat{i} + 2\hat{j} + 2\hat{k} \) and \( \vec{d} = -8\hat{i} + 5\hat{j} + 3\hat{k} \).

(i) Determine scalars \( \lambda, \mu \) and \( \eta \), such that

\[
\vec{d} = \lambda \vec{a} + \mu \vec{b} + \eta \vec{c}.
\]

(This shows that \( \vec{d} \) can be expressed as a linear combination of vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \), that is, it is linearly dependent on these three vectors.)

(ii) Why does the construction fail if \( \vec{c} = -5\hat{i} + 5\hat{j} - 3\hat{k} \)?
Solution.

(i) The scalars $\lambda$, $\mu$ and $\eta$ must satisfy the equation

$$\vec{d} = \lambda \vec{a} + \mu \vec{b} + \eta \vec{c}. $$

This one vector equation must be solved for the three unknown scalars. It states that

$$-8 \hat{i} + \hat{j} + \hat{k} = \lambda (2 \hat{i} + \hat{j}) + \mu (3 \hat{i} - \hat{j} + \hat{k}) + \eta (-\hat{i} + 2 \hat{j} + 2 \hat{k}).$$

After using the rules of vector addition and scalar multiplication and rearranging the terms onto the left hand side of the equality we obtain:

$$(-8 - 2\lambda - 3\mu + \eta) \hat{i} + (1 - \lambda + \mu - 2\eta) \hat{j} + (1 - \mu - 2\eta) \hat{k} = \emptyset.$$ 

Now the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are linearly independent. Why? So the only way a linearly combination of these vectors can equal $\emptyset$ is for the coefficients to be all zero, that is:

$$-8 - 2\lambda - 3\mu + \eta = 0,$$

$$1 - \lambda + \mu - 2\eta = 0,$$

$$1 - \mu - 2\eta = 0.$$ 

This set of linear equations is easily solved by elimination methods, eliminating one variable at a time, yielding the unique solution

$$\lambda = -2, \mu = -1, \text{ and } \eta = 1.$$ 

(For more precise details of solving linear equations see Chapter 3.) Hence we have that

$$\vec{d} = -2 \vec{a} - \vec{b} + \vec{c}.$$ 

(ii) Let us now consider the new vector $\vec{c} = -5 \hat{i} + 5 \hat{j} - 3 \hat{k}$. Following the analysis given in (i) we arrive at the following set of linear equations

$$-8 - 2\lambda - 3\mu + 5\eta = 0,$$

$$1 - \lambda + \mu - 5\eta = 0,$$

$$1 - \mu + 3\eta = 0.$$ 

for the three scalars $\lambda$, $\mu$ and $\eta$. Verify these calculations. These set of linear equations are inconsistent, that is, they have no solution. We can see this as follows. Take $2 \times$ second equation from the first equation. We obtain

$$-10 - 5\mu + 15\eta = 0 \text{ or } -2 - \mu + 3\eta = 0.$$ 

Clearly this equation is not in agreement with the third equation which says that $-\mu + 3\eta = -1$. There can be no solution to the set of equations.
Why does the construction fail in this case? With this new vector \( \mathbf{c} \) the set \( \{ \mathbf{a}, \mathbf{b}, \mathbf{c} \} \) is not linearly independent. Why? Because \( \mathbf{c} \) lies in the plane spanned by the vectors \( \mathbf{a} \) and \( \mathbf{b} \). You can easily verify that in this case \( \mathbf{c} = 2\mathbf{a} - 3\mathbf{b} \). They do not form a basis for \( \mathbb{R}^3 \).

However in case (i), the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) form a linearly independent set and so a basis for \( \mathbb{R}^3 \). See if you can verify this directly from the definition.

\[ \square \]

### 2.4.1 Parametrised equation of a plane

In this section we will examine the parametric equation of a plane in \( \mathbb{R}^3 \).

Suppose we consider the plane through a point \( C \) parallel to two independent vectors \( \mathbf{a} \) and \( \mathbf{b} \), see Figure 2.14. Let \( \mathbf{c} \) be the position vector of the point \( C \) relative to some given origin \( O \).

![Figure 2.14: Plane through \( C \) parallel to \( \mathbf{a} \) and \( \mathbf{b} \)](image)

Let \( \mathbf{r} \) be the position vector to any point \( P \) lying on the plane here spanned by the vectors \( \mathbf{a} \) and \( \mathbf{b} \). Now from previous arguments we know there must exist two scalars \( s \) and \( t \) such that the vector

\[
\mathbf{CP} = s\mathbf{a} + t\mathbf{b}.
\]

By the triangle law the position vector of \( P \) is given by

\[
\mathbf{r}(s, t) = \mathbf{c} + \mathbf{CP} = \mathbf{c} + s\mathbf{a} + t\mathbf{b} \quad \text{where} \quad s \quad \text{and} \quad t \quad \text{are scalars.} \tag{2.7}
\]

We write the position vector to \( P \) as \( \mathbf{r}(s, t) \) to indicate that this position vector is a vector function of two arguments \( s \) and \( t \). As \( s \) and \( t \) take real values the locus of all points \( P \) forms the plane through \( C \) parallel to the two vectors \( \mathbf{a} \) and \( \mathbf{b} \).
Equation 2.7 is called the parametric equation of this plane. The two parameters are the two scalars $s$ and $t$.

**EXAMPLE 2.7**

Find the parametric equation of the plane passing through three points $A$, $B$ and $C$, the points not being collinear, with position vectors relative to a given origin $O$, being $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

**Solution.** Consider Figure 2.15 which depicts the plane through points $A$, $B$ and $C$. Clearly the vectors $\vec{AB} = \vec{b} - \vec{a}$ and $\vec{AC} = \vec{c} - \vec{a}$ are linearly independent, since the points are not collinear. They form a basis for the plane and so are a spanning set. Therefore for the vector $\vec{AP}$ lying in the plane there exist scalars $s$ and $t$, such that

$$\vec{AP} = s\vec{AB} + t\vec{AC} = s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a}).$$

Thus any point $P$ on the plane has position vector $\vec{r}$ relative to the origin

$$\begin{align*}
\vec{r}(s,t) &= \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{b}) \\
&= (1 - s - t)\vec{a} + s\vec{b} + t\vec{c}.
\end{align*}$$

Again we indicate that $\vec{r}$ is a vector function of two parameters $s$ and $t$. This equation defines the parametric equation of the desired plane.

[Box]
CHAPTER 2. VECTORS

Now to tackle an example using the Cartesian coordinate representation.

EXAMPLE 2.8
Suppose we are given three points \( A \), \( B \) and \( C \) with position vectors \( \vec{a} = i \), \( \vec{b} = j \) and \( \vec{c} = k \) relative to a given origin \( O \).

(a) Determine the parametric equation of the plane passing through these three points.

(b) Does the point \( P \) with position vector \( \vec{p} = -(1/4)i + (1/2)j + (3/4)k \) lie on the plane defined in (a)?

Solution.
(a) The desired parametric equation of the plane through points \( A \), \( B \) and \( C \) is obtained by Equation 2.8. So the position vector \( \vec{r} \) of any point \( P \) on the plane is

\[
\vec{r}(s, t) = (1 - s - t)i + s j + t k,
\]

where \( s \) and \( t \) are arbitrary scalars.

(b) For the desired point \( P \) with position vector \( \vec{p} \) to lie on this plane there must exist values of the scalars \( s \) and \( t \) such that

\[
\vec{p} = (1 - s - t)i + s j + t k = -(1/4)i + (1/2)j + (3/4)k.
\] (2.9)

We can now argue two ways to find the values of \( s \) and \( t \) if they exist so that this vector equality is satisfied.

(i) We know that for the basis \( \{i, j, k\} \) the coordinate representation for a point is unique. This means by equating the corresponding coordinates for the given basis axes, that the following three equations are satisfied:

\[
1 - s - t = -1/4, \\
\frac{1}{2} = \frac{1}{2}, \\
t = \frac{3}{4}.
\]

These three linear equations for the two unknowns are consistent with an easy determined solution, namely \( s = 1/2 \) and \( t = 3/4 \). We conclude that the point \( P \) must lie on the plane. If these equations had been inconsistent with no solution for \( s \) and \( t \), we would conclude that the point \( P \) did not lie on the plane.

(ii) The second approach uses the linear independence of the basis vectors \( i \), \( j \) and \( k \). Rearranging terms in Equation 2.9 we obtain

\[
(1 - s - t + 1/4)i + (s - 1/2)j + (t - 3/4)k = 0.
\]
Now the set \( \{ \hat{i}, \hat{j}, \hat{k} \} \) is a basis, the vectors are linearly independent and there is no linear combination of these vectors that sum to the zero vector unless all the coefficients of the vectors are zero, that is:

\[
1 - s - t + 1/4 = 0, \quad s - 1/2 = 0, \quad t - 3/4 = 0.
\]

The set of linear equations is the same as obtained in (i), and we continue with the analysis to the same conclusion.

\[\square\]

**EXERCISE 2.3**

Show that the point \( P \) with position vector \( \vec{p} = \hat{i} + \hat{j} + 2\hat{k} \) does not lie on the plane defined in Example 2.8.

### 2.4.2 Cartesian equation of a line in \( \mathbb{R}^3 \)

Now that we have a Cartesian coordinate representation for points in space, we can find the cartesian equation of a straight line in \( \mathbb{R}^3 \). It is quite different from that which you know as the cartesian equation of a straight line on a plane.

Suppose we are given the points \( A, B \) and \( C \) with position vectors \( \vec{a} = \hat{i} - \hat{j} + 6\hat{k}, \vec{b} = 2\hat{i} + \hat{j} \) and \( \vec{c} = -3\hat{i} + 2\hat{j} - 4\hat{k} \) respectively, relative to an origin \( O \).

What is the parametric equation of the line passing through \( A \) parallel to the line segment \( BC \)?

We refer the reader back to Example 2.1 and Equation 2.1. The direction of the line is specified by the vector \( \vec{d} = \vec{BC} = \vec{c} - \vec{b} \). Write

\[
\vec{d} = \vec{BC} = \vec{c} - \vec{b} = (-3\hat{i} + 2\hat{j} - 4\hat{k}) - (2\hat{i} + \hat{j}) = -5\hat{i} + \hat{j} - 4\hat{k}.
\]

The required line is parallel to this vector. Hence by Equation 2.1 we find that the position vector \( \vec{r} \) to any point \( P \) on the line is given by

\[
\vec{r}(t) = \vec{a} + t\vec{d} = (\hat{i} - \hat{j} + 6\hat{k}) + t(-5\hat{i} + \hat{j} - 4\hat{k}) = (1 - 5t)\hat{i} + (t - 1)\hat{j} + (6 - 4t)\hat{k}.
\]

(2.10)

where scalar \( t \) is an arbitrary parameter. This Equation 2.10 is the vector parametric equation of the desired line through \( A \) and parallel to the line segment \( BC \).
The position vector to any point $P$ on the line is a function of the scalar parameter $t$. Now if we set

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k},$$

we are determining the coordinates of the point $P$ given by $\vec{r}(t)$ to be $(x(t), y(t), z(t))$. Notice the coordinates are functions of the parameter $t$.

Since the coordinate representation with respect to the given basis $\{\hat{i}, \hat{j}, \hat{k}\}$ is unique, we can equate respective coordinates on both sides of Equation 2.10. From this we obtain:

$$x(t) = 1 - 5t, \quad y(t) = t - 1, \quad z(t) = 6 - 4t.$$  

Eliminating $t$ from these three equations we establish

$$\frac{x - 1}{-5} = \frac{y + 1}{1} = \frac{z - 6}{-4} = t \text{ where } t \text{ is arbitrary.} \quad (2.11)$$

This is the Cartesian equation of the required line in $\mathbb{R}^3$.

It is not too difficult to follow through with a general argument. This is left as an Exercise.

**EXERCISE 2.4**

Show that the cartesian equation of the line passing through the point $A$ with position vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ relative to an origin $O$, and parallel to the vector $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by the equation

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}. \quad (2.12)$$

**REMARK 2.3**

Observe that the Cartesian Equation 2.12 consists of three equations, namely:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}.$$  

Indeed we see from Equation 2.11 that each term equals the arbitrary parameter used to describe the line. This fact can allow us to retrieve the parametric equation of the line, given its Cartesian equation.
EXERCISE 2.5

Given the following Cartesian equation of a line in $\mathbb{R}^3$

\[
\frac{x - 1}{2} = \frac{y - 3}{-2} = \frac{z + 3}{3},
\]

determine the parametric equation of the line and define a point on the line and a vector specifying its line of action.

EXAMPLE 2.9

Determine if the points $Q_1$ and $Q_2$ with position vectors $q_1 = -9i + j - 2k$ and $q_2 = -9i + 2j - 2k$ respectively, lie on the line described by the parametric Equation 2.11

\[
\frac{x - 1}{-5} = \frac{y + 1}{1} = \frac{z - 6}{-4}.
\]

Solution. To determine whether the point $Q_1$ lies on this line we must show that the equation is satisfied for $x = -9$, $y = 1$, and $z = -2$. Since

\[
\frac{-9 - 1}{-5} = \frac{1 + 1}{1} = \frac{-2 - 6}{-4} = 2,
\]

it is true. The point does lie on the line and the value of the parameter is $t = 2$.

Now examining in the same manner the point $Q_2$, we find

\[
\frac{-9 - 1}{-5} = 2, \quad \frac{2 + 1}{1} = 3, \quad \text{and} \quad \frac{-2 - 6}{-4} = 2.
\]

The equation is not satisfied by the coordinates of $Q_2$, so we conclude the $Q_2$ does not lie on the given line.

\[
\square
\]

2.5 The Dot (Scalar or Inner) Product

In this section we define the dot product of two vectors and discuss its properties. Sometimes in the Algebra literature this dot product is called the scalar product since the product of the two vectors produces a scalar output. It is also referred to in the literature as an inner product. You might correctly suspect there is therefore such a quantity as an outer product and this is true. Using only the geometric definition of a vector initially given we make the following definition.
DEFINITION 2.9

The dot product of two vectors \( \vec{a} \) and \( \vec{b} \) is the real number \( |a||b| \cos(\theta) \) where \( \theta \in [0, \pi] \) is the angle between the directions of the vectors \( \vec{a} \) and \( \vec{b} \). We write

\[ \vec{a} \cdot \vec{b} = |a||b| \cos(\theta) = ab \cos(\theta). \]

The scalar product is read “\( \vec{a} \) dot \( \vec{b} \)”.

Note that the angle is restricted to lie in the interval \([0, \pi]\) and \( a = |a| \) and \( b = |b| \).

Figure 2.16: The scalar product of two vectors

The dot product introduces us to a new binary operator which we call “dot” that acts upon two vectors to produce a new scalar quantity called the dot or scalar product of the two vectors. (Note that the angle is restricted to \([0, \pi]\) in the definition and that we have used the notation \( a \) to be the magnitude of the vector \( \vec{a} \).) It is important to write vector notation correctly. Clearly \( a = |a| \) is a scalar and cannot be equated with the vector \( \vec{a} \) which has both magnitude and direction.

2.5.1 Properties of the dot product

Let us now examine some important properties of this new operator.

SP1: The dot product \( \vec{a} \cdot \vec{b} \) is a SCALAR quantity.

SP2: (Commutative Law) Given any two vectors \( \vec{a} \) and \( \vec{b} \) then

\[ \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}. \]

This is the commutative law and indicates that the taking of a dot product of two vectors is independent of the order in which the dot product is taken.

SP3: Given vector \( \vec{a} \) and unit vector \( \hat{n} \) then the projection of \( \vec{a} \) in the direction of the vector \( \hat{n} \) is the vector

\[ \vec{a}_p = (\vec{a} \cdot \hat{n})\hat{n}. \]
The length of the projection is

\[ |a_p| = a \cdot \hat{n} = |a| \cos(\theta) = a \cos(\theta), \]

where \( \theta \in [0, \pi] \) is the included angle between the two directions specified by the two given vectors.

**SP4:** It is easy to establish that

\[ a \cdot b = |a||b| \cos(\theta) \]
\[ = ab \cos(\theta) \]
\[ = (\text{length of } a)(\text{length } b_p) \]

where \( a = |a| \) and \( b = |b| \) are the magnitudes of the two vectors.

**Figure 2.17:** Diagram for SP4

**SP5:** Given any two vectors \( \sim{a} \) and \( \sim{b} \) then

If \( a \cdot b = 0 \), then either

(i) \( a \perp b \),

or

(ii) \( |a| = 0 \),

or

(iii) \( |b| = 0 \).

This means that if the dot product of two vectors is zero, then the following cases may occur; either the two vectors are perpendicular provided they are NOT the zero vector, or one of the two vectors or both, must be the zero vector as it must have zero magnitude.

Clearly if we take the dot product of two non-zero vectors whose directions are perpendicular to each other then the resulting dot product will be zero since \( \cos(\pi/2) = 0 \). If either of the two vectors is the zero vector then the dot product will also be zero.

**SP6:** Suppose we are given two vectors \( a \) and \( b \). Then if the two vectors are parallel (we write \( a \parallel b \)), then \( a \cdot b = |a||b| \) if the vectors have the same sense, that is \( \theta = 0 \), or \( a \cdot b = -|a||b| \) if the vectors have the opposite sense, that is, \( \theta = \pi \).

**SP7:** Given any two vectors \( a \) and \( b \) and any two scalars \( \lambda \) and \( \mu \) then it can be shown that

\[ (\lambda a) \cdot (\mu b) = (\lambda \mu) a \cdot b. \]
CHAPTER 2. VECTORS

The proof of SP7 can be completed in a straightforward manner using geometry, the definition of the vector and the dot product. It is left as an exercise. Note that three cases must be discussed (i) \( \lambda, \mu \geq 0 \), (ii) \( \lambda, \mu \leq 0 \) and (iii) one of \( \lambda \) or \( \mu \) is negative, ignoring the trivial case when both scalars are zero.

**SP8:** (Distributive law) Given any three vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) then

\[
\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.
\]

This distributive law says that “the dot product is distributive over vector addition”. It is in fact the left distributive law. The right distributive law

\[
(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a},
\]

is the same since the dot product is commutative. The proof of this law can be shown geometrically, and is shown in a latter example.

### 2.5.2 Coordinate representation of the dot product

Consider the particular rectangular basis vectors \( \vec{i}, \vec{j} \) and \( \vec{k} \) that we have chosen to represent a vector in \( \mathbb{R}^3 \). From the previous definition of a unit vector and the definition of the dot product we can establish the following Table 2.1.

<table>
<thead>
<tr>
<th>Basis Vector</th>
<th>( \vec{i} \cdot \vec{i} )</th>
<th>( \vec{j} \cdot \vec{j} )</th>
<th>( \vec{k} \cdot \vec{k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{i} \cdot \vec{i} )</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \vec{j} \cdot \vec{j} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \vec{k} \cdot \vec{k} )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table 2.1: Dot product basis table**

Using this information and some of the properties of the dot product we can find a formula for the dot product of any two vectors in terms of their coordinate representation. Suppose we have two vectors \( \vec{a} \) and \( \vec{b} \) with coordinate representation in terms of the basis vectors:

\[
\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k} \quad \text{and} \quad \vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}.
\]

Then

\[
\vec{a} \cdot \vec{b} = a \cdot (b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k})
\]

\[
= a \cdot (b_1 \vec{i} + \vec{c}) \quad \text{where} \quad \vec{c} = b_2 \vec{j} + b_3 \vec{k}
\]

\[
= a \cdot (b_1 \vec{i}) + a \cdot \vec{c} \quad \text{(Distributive law)}
\]

**Note:** The example of SP8 is shown geometrically, and is shown in a former example.
2.5. THE DOT (SCALAR OR INNER) PRODUCT

\[ a \cdot \hat{a} = a_1 \hat{a}_1 + a_2 \hat{a}_2 + a_3 \hat{a}_3 \]

This formula should bring to memory that derived for the product of a row vector and a column vector in the previous chapter on matrices. Using the convention that vectors are represented as column vectors and writing

\[ \hat{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{and} \quad \hat{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \]

In summary given two vectors \( \hat{a} \) and \( \hat{b} \) with coordinate representation

\[ \hat{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k} \quad \text{and} \quad \hat{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}. \]

then

\[ \hat{a} \cdot \hat{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{k=1}^{3} a_k b_k. \] (2.13)
then the dot product can be written in matrix notation
\[ \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{k=1}^{3} a_k b_k. \]

It is now clear that for any vector \( \mathbf{a} \)
\[ \mathbf{a} \cdot \mathbf{a} = \mathbf{a}^T \mathbf{a} = |\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2 = \sum_{k=1}^{3} a_k^2, \quad (2.14) \]
which is consistent with the geometrical construction using the basis vectors, that gives the magnitude of the vector using the Pythagoras Theorem.

**EXAMPLE 2.10**
*Use the dot product to find the angle between the vectors*

\[ \mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}. \]

**Solution.** Let the required angle be \( \theta \in [0, \pi] \). By the definitions and properties shown above we have
\[ ab \cos(\theta) = \mathbf{a} \cdot \mathbf{b} = (6)(2) + (-3)(2) + (1)(-1) = 5. \quad (2.15) \]
Now the magnitudes of the two vectors are:
\[ a = |\mathbf{a}| = ((6)^2 + (-3)^2 + (1)^2)^{1/2} = (46)^{1/2}, \]
\[ b = |\mathbf{b}| = ((2)^2 + (2)^2 + (-1)^2)^{1/2} = (9)^{1/2} = 3. \]

From Equation 2.15, we find
\[ \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{5}{3\sqrt{46}}. \]
Using the inverse cosine function, we calculate that the required angle is \( \theta \approx 75^\circ 46' \) or \( \theta \approx 75.775^\circ \).
2.5. THE DOT (SCALAR OR INNER) PRODUCT

In this last example we see how easy it is to calculate the length of a vector and to find the angle between two vector directions when given their coordinate representation relative to a given basis such as \( \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \). For our original definition of a vector being basically a directed line segment, this was not so easy. In that definition the concept of length and angle were assumed inherent in the geometry of the Euclidean space of vectors.

**EXAMPLE 2.11**

(a) Find the length and direction cosines of the vector \( \mathbf{a} \) from the point \((1, -1, 3)\) to the midpoint of the line segment from the origin to the point \((6, -6, 4)\).

(b) Find the dot (scalar) product of the two vectors \( \mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} \) and \( \mathbf{b} = \mathbf{i} + \mathbf{j} + 4\mathbf{k} \).

(c) Find the length of the projection of the vector \( 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \) on the vector \( \mathbf{i} + \mathbf{j} + 2\mathbf{k} \).

(d) What value of the scalar \( \lambda \) makes the vector \( 2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k} \) and \( 4\mathbf{i} - 2\mathbf{j} - 2\lambda\mathbf{k} \) perpendicular?

(e) Using vector methods and the dot product, show that the median of an isosceles triangle is perpendicular to its base, see Figure 2.18.

**Solution.**

(a) Let \( M \) be the midpoint of the line segment joining the origin to \( 6\mathbf{i} - 6\mathbf{j} + 4\mathbf{k} \). The position vector of \( M \),
\[
\mathbf{m} = (6\mathbf{i} - 6\mathbf{j} + 4\mathbf{k})/2 = 3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.
\]
Hence the vector \( \mathbf{a} = (3\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) - (\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} \).
The length of \( \mathbf{a} \),
\[
|\mathbf{a}| = \left( (2)^2 + (-2)^2 + (-1)^2 \right)^{1/2} = 3.
\]
The direction cosines of \( \mathbf{a} \) are
\[
\gamma_1 = 2/3, \quad \gamma_2 = -2/3 \quad \text{and} \quad \gamma_3 = -1/3.
\]

(b) The dot product of the two vectors is given by:
\[
\mathbf{a} \cdot \mathbf{b} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = (2)(1) + (1)(1) + (-3)(4) = -9.
\]
(c) Let \( \mathbf{a} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} \) and \( \mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} \). The length of the projection of \( \mathbf{a} \) on \( \mathbf{b} \) is
\[
\mathbf{a} \cdot \hat{\mathbf{b}} = \frac{\mathbf{a} \cdot (\mathbf{b}/|\mathbf{b}|)}{|\mathbf{b}|} = \frac{(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{|\mathbf{i} + \mathbf{j} + 2\mathbf{k}|} = \frac{(6)(1) + (3)(1) + (2)(2)}{\sqrt{6}} = \frac{13}{\sqrt{6}}.
\]

(d) As each vector is not the zero vector, vector \( 2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k} \) will be perpendicular to \( 4\mathbf{i} - 2\mathbf{j} - 2\lambda\mathbf{k} \) when
\[
(2\mathbf{i} + \lambda\mathbf{j} + \mathbf{k}) \cdot (4\mathbf{i} - 2\mathbf{j} - 2\lambda\mathbf{k}) = 0.
\]
This implies that
\[
(2)(4) + (-2)\lambda + (-2)\lambda = 0, \quad \text{and} \quad \lambda = 2.
\]
Hence the required value to make the vectors perpendicular is \( \lambda = 2 \).

(e) Let the isosceles triangle \( \triangle OAB \) with equal sides \( OA \) and \( OB \) be as shown in Figure 2.18. The origin has been selected as one of the vertices for convenience. The triangle is then formed by the position vectors of points \( A \) and \( B \), \( \mathbf{a} \) and \( \mathbf{b} \) respectively. Since \( OA = OB \), \( |\mathbf{a}| = |\mathbf{b}| \). The median joining \( O \) to the midpoint of \( AB \) is \( \mathbf{OD} \). We are required to show that \( \mathbf{OD} \) is perpendicular to the segment \( AB \). Now \( \mathbf{OD} = (\mathbf{a} + \mathbf{b})/2 \) since \( D \) is the midpoint of the segment \( AB \). Further, \( \mathbf{AB} = \mathbf{b} - \mathbf{a} \). These are both non-zero vectors, otherwise the isosceles triangle would not exist. Therefore
\[
\mathbf{OD} \cdot \mathbf{AB} = \frac{(\mathbf{a} + \mathbf{b})}{2} \cdot (\mathbf{b} - \mathbf{a})
\]
\[ (1/2) \left( a \cdot b + b \cdot b - b \cdot a - a \cdot a \right) \]
\[ = (1/2) \left( |b|^2 - |a|^2 \right) \]

2.5.3 Cartesian equation of a plane

We begin this section by describing the normal from of the equation of a plane in \( \mathbb{R}^3 \) by way of an example.

**EXAMPLE 2.12**

(a) Show that the vector equation of a plane may be written in the scalar product or normal form

\[ \mathbf{r} \cdot \mathbf{\hat{n}} = q, \]

where \( \mathbf{\hat{n}} \) is a unit vector normal, that is, perpendicular to the plane and \( q \) is a constant.

(b) Show that the perpendicular distance from the origin to the plane is \( |q| \).

**Solution.** Let \( Q \) be the foot of the perpendicular from the origin to the plane as shown in Figure 2.19. Assume that \( \mathbf{\hat{n}} \) is a unit vector along \( OQ \) from \( O \). If \( P \) is any point in the plane given by the position vector \( \mathbf{r} \) relative to the origin \( O \), then \( \mathbf{QP} \) must be perpendicular to \( \mathbf{\hat{n}} \) always.

Let \( p = ON \) be the perpendicular distance to the plane from \( O \). Now

\[ \mathbf{OQ} = p\mathbf{\hat{n}} \quad \text{and} \quad \mathbf{QP} = \mathbf{r} - p\mathbf{\hat{n}}. \]

So for \( \mathbf{QP} \) to be perpendicular to \( \mathbf{\hat{n}} \) we require

\[ \mathbf{QP} \cdot \mathbf{\hat{n}} = 0 \quad \text{or} \quad (\mathbf{r} - p\mathbf{\hat{n}}) \cdot \mathbf{\hat{n}} = 0. \]
From this last equation we achieve the required normal form for the equation of the plane, namely:

\[ \vec{r} \cdot \hat{n} = p. \]  \hspace{1cm} (2.16)

It should be remarked that in Figure 2.19 we have indicated that the line of action for \( \hat{n} \) is from the origin to the plane. Now for any given plane there can be two vectors which are normal to it, namely \( \hat{n} \) and \(-\hat{n}\). If the normal we had chosen pointed from \( Q \) to \( O \), then the unit vector to the plane is \(-\hat{n}\), that is, \( O \) lies on the opposite side of the plane to that shown in the figure, and we would have derived the equation:

\[ \vec{r} \cdot \hat{n} = -p. \]  \hspace{1cm} (2.17)

Hence the equation of the plane in general, may be written as

\[ \vec{r} \cdot \hat{n} = q, \]  \hspace{1cm} (2.18)

where \( |q| \) is the perpendicular distance from the origin \( O \) to the plane. The required results in (a) and (b) are now established.

For a given plane let the normal \( \hat{n} \) to the plane be given in coordinate form as

\[ \hat{n} = n_1 \hat{i} + n_2 \hat{j} + n_3 \hat{k}. \]

Since \( \hat{n} \) is a unit vector then \( n_1^2 + n_2^2 + n_3^2 = 1 \).

Let the position vector of any point in the plane \( \vec{r} \) be similarly represented as

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}. \]
2.5. THE DOT (SCALAR OR INNER) PRODUCT

The point \( P \) has coordinates \((x, y, z)\). Using Equation 2.18 and these representations the equation of the plane is given by

\[
r \cdot \hat{n} = n_1x + n_2y + n_3z = q,
\]

(2.19)

where \(|q|\) is the perpendicular distance from the origin to the plane.

In general, given scalars \(a, b\) and \(c\), the equation

\[
a x + b y + c z = d,
\]

(2.20)

represents the locus of points \((x, y, z)\) on a plane in \(\mathbb{R}^3\). Equation 2.20 is called the Cartesian form of the equation to a plane.

This equation is not in the same form as Equation 2.19, since \(a^2 + b^2 + c^2 \neq 1\). It may be converted into the form of Equation 2.19, by dividing by \(m = (a^2 + b^2 + c^2)^{1/2}\) to obtain

\[
\frac{a}{m}x + \frac{b}{m}y + \frac{c}{m}z = \frac{d}{m},
\]

or

\[
n_1x + n_2y + n_3z = q,
\]

where \(n_1 = a/m, \ n_2 = b/m, \ n_3 = c/m\) and \(q = d/m\). Now we have \(n_1^2 + n_2^2 + n_3^2 = 1\) and the equation is in the required form. The transformation tells us that Equation 2.20 is a plane,

\[
\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k},
\]

\[
= \frac{a}{m}\hat{i} + \frac{b}{m}\hat{j} + \frac{c}{m}\hat{k},
\]

is a vector perpendicular to the plane, and furthermore:

(i) If \(q = d/m > 0\), then \(|q|\) is the perpendicular distance from the origin to the plane and \(\hat{n}\) points from the origin to the plane.

(ii) If \(q = d/m < 0\), then \(\hat{n}\) points away from the origin to the plane.

**REMARK 2.4**

It should now be clear with this information that if we are given the equation

\[
a x + b y + c z = d,
\]

we can physically picture this plane with reference to the set of coordinate axes \(\{\hat{i}, \hat{j}, \hat{k}\}\).
EXERCISE 2.6
Consider the plane described by the Cartesian equation
\[ x + y + z = 1. \]

(i) Determine a unit vector normal to the plane.

(ii) Determine the perpendicular distance from the origin to the plane.

(iii) Show that this plane is the same plane discussed in Example 2.8.

The next illustration shows us how to find the normal form of the equation of a plane which has normal \( \vec{n} \), and passes through a specified point \( P \).

ILLUSTRATION 2.1
Suppose we are given a plane passing through the point \( A \) with position vector \( \vec{a} \) relative to origin \( O \), and with normal \( \vec{n} \). Determine the equation of the plane in normal form.

Solution. Let \( P \) be any point in the plane with position vector \( \vec{r} \) relative to the origin, see Figure 2.20. Now the vector \( \vec{AP} = \vec{r} - \vec{a} \) must always be perpendicular to the vector \( \vec{n} \) for any such point \( P \) in the plane. This condition requires:

\[ \vec{AP} \cdot \vec{n} = (\vec{r} - \vec{a}) \cdot \vec{n} = 0. \]

Hence the equation to the plane in normal form is

\[ \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}. \] (2.21)
2.5. THE DOT (SCALAR OR INNER) PRODUCT

The illustration above was done in general terms. Here is an exercise for you to work through using the Cartesian representation for the vectors.

**EXERCISE 2.7**

*Using the result from Illustration 2.1, namely Equation 2.21, determine the Cartesian equation to the plane with normal vector \( \vec{n} = \vec{i} + 2\vec{j} + 3\vec{k} \) and passing through the point \( A \) which has position vector \( \vec{a} = \vec{i} + \vec{j} + \vec{k} \) relative to the origin. What is the perpendicular distance from the origin to this plane?*

In the next illustration we show how to prove the distributive law SP8 by using the simple geometric arguments behind the definition of the dot product.

**ILLUSTRATION 2.2** *Prove that the dot (scalar) product is distributive over vector addition*

\[
\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.
\]

**Solution.** We construct the following geometry shown in Figure 2.21. With origin

![Figure 2.21: Figure for the proof of the distributive law SP8](image)

\( O \) as selected, form triangle \( OBC \) formed by vectors \( \vec{b} \) and \( \vec{c} \) such that \( OC = \vec{b} + \vec{c} \). Drop perpendicular \( CC'' \) from \( C \) to the line with direction of \( \vec{a} \) passing through \( O \). Similarly drop perpendicular \( BB' \) from \( B \) to this line. Form the line segment \( BC'' \)
through $B$ parallel to the vector $\mathbf{a}$. Then by the previous properties of the dot product we have

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}|OC' \quad \text{(Property SP4)}$$

$$= |\mathbf{a}|(OB' + BC'')$$

$$= |\mathbf{a}|(OB' + BC'')$$

$$= |\mathbf{a}| \left( \mathbf{b} \cdot \frac{a}{|\mathbf{a}|} + \mathbf{c} \cdot \frac{a}{|\mathbf{a}|} \right) \quad \text{(Property SP3)}$$

$$= |\mathbf{a}| \left( \frac{1}{|\mathbf{a}|}(\mathbf{b} \cdot \mathbf{a}) + \frac{1}{|\mathbf{a}|}(\mathbf{c} \cdot \mathbf{a}) \right) \quad \text{(Property SP7)}$$

$$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \quad \text{(Property SP2)}.$$

\[\square\]

### 2.6 The vector (cross) product

Having completed our analysis of the dot product, in this section we define a new binary operator called the vector or cross product operator, which acts on two given vectors to form a new vector.

**DEFINITION 2.10**

The vector product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \mathbf{\hat{n}} = ab \sin(\theta) \mathbf{\hat{n}}.$$

The vector product is read “$\mathbf{a}$ cross $\mathbf{b}$” and is a vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ (imagined to be drawn from a common point), defined in the right hand sense in going from $\mathbf{a}$ and $\mathbf{b}$. It has magnitude $ab \sin(\theta)$ where $a = |\mathbf{a}|$, $b = |\mathbf{b}|$ and $\theta \in [0, \pi]$ is the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$.

![Figure 2.22: The vector product of two vectors](image)
The unit vector \( \hat{n} \) is of course perpendicular to the plane of \( \hat{a} \) and \( \hat{b} \). It is important to note that

The sense of direction of \( \hat{n} \) is chosen so that a right hand screw perpendicular to the plane of \( \hat{a} \) and \( \hat{b} \) would move in the direction \( \hat{n} \) if rotated in the sense that takes \( \hat{a} \) into \( \hat{b} \) through an angle \( \theta \).

The vector product is also commonly called the cross product in the literature, for obvious reasons. You would be well advised to ensure that you write this vector as a vector NOT a scalar. The vector product is completely different from the dot or scalar product. The definition has been given using the basic geometric definition of a vector. We will move onto its definition in terms of coordinates in the \( \{i, j, k\} \) basis system shortly.

2.6.1 Properties of the vector product

Let us now examine some properties of this important operator.

**VP1:** The vector product \( \hat{a} \times \hat{b} \) is a VECTOR quantity and should be written correctly as a vector in all mathematical analysis.

**VP2:** Given any two vectors \( \hat{a} \) and \( \hat{b} \) then

\[
\hat{a} \times \hat{b} = -(\hat{b} \times \hat{a}).
\]

This result follows from the fact that interchanging the order to the two vectors changes the sense of direction to the unit vector \( \hat{n} \). Hence the vector product does not obey the commutative law as does the dot product. The order in which the two vectors are taken in the vector product is extremely important otherwise the wrong answer will be obtained.

**VP3:** Given any two vectors \( \hat{a} \) and \( \hat{b} \) then

\[
\text{If } \hat{a} \times \hat{b} = \emptyset, \text{ then either } (i) \quad \theta = 0 \text{ or } \pi, (a \parallel b), \\
\text{or } (ii) \quad |a| = 0, \\
\text{or } (iii) \quad |b| = 0.
\]

This means that if the vector product of two vectors is the zero vector, then the following cases may occur; either the two vectors are parallel provided they are NOT the zero vector, or one of the two vectors or both, must be the zero vector as it must
have zero magnitude. If the two vectors are parallel then clearly the cross product \( \vec{a} \times \vec{b} = \vec{0} \).

**VP4:** Given any two vectors \( \vec{a} \) and \( \vec{b} \) and any two scalars \( \lambda \) and \( \mu \) then it can be shown that
\[
(\lambda \vec{a}) \times (\mu \vec{b}) = (\lambda \mu) \vec{a} \times \vec{b}.
\]
The proof of VP4 can be completed in a straightforward manner using geometry as in the case of the scalar product. It is left as an exercise. Note again that three cases must be discussed (i) \( \lambda, \mu \geq 0 \), (ii) \( \lambda, \mu \leq 0 \) and (iii) one of \( \lambda \) or \( \mu \) is negative, ignoring the trivial case when both scalars are zero. In brief this property says that scalars may be taken outside the vector product, so for example:
\[
(3 \vec{a}) \times (-\vec{b}) = (3)(-1)(\vec{a} \times \vec{b}) = -3(\vec{a} \times \vec{b}).
\]

**VP5:** (Distributive laws) Given any three vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) then

(i) Left distributive law
\[
\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.
\] (2.22)

(ii) Right distributive law
\[
(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}
\] (2.23)

This distributive laws say that “the vector product is distributive over vector addition”. There are two such laws, both a left and right distributive law since the vector product in not commutative. The proof of the left distributive law is given at the end of this section.

**VP6** Given any three vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) then the vectors \( \vec{a} \times (\vec{b} \times \vec{c}) \) and \( (\vec{a} \times \vec{b}) \times \vec{c} \) are well defined. However the vectors are NOT EQUAL, that is,
\[
\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.
\]

Why is this so? It suffices to produce one set of vectors for which the two vectors are not equal. Using the Cartesian basis vectors, take \( \vec{a} = \vec{b} = \vec{i} \) and \( \vec{c} = \vec{j} \). Then
\[
\vec{b} \times \vec{c} = \vec{i} \times \vec{j} = |\vec{i}||\vec{j}| \sin(\pi/2) \vec{k} = \vec{k},
\]
since both basis vectors are unit vectors and clearly \( \vec{k} \) is a unit vector perpendicular to the plane of \( \vec{i} \) and \( \vec{j} \) taken in the right hand sense. This means that
\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = |\vec{i}||\vec{k}| \sin(\pi/2)(-\vec{j}) = -\vec{j},
\]

\[
(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{i} \times \vec{i}) \times \vec{j} = \vec{k} \times \vec{j} = |\vec{k}||\vec{j}| \sin(\pi/2) \vec{i} = \vec{i}.
\]
2.6. THE VECTOR (CROSS) PRODUCT

since in this case \( \mathbf{j} \) is a unit vector perpendicular to the plane of \( \mathbf{i} \) and \( \mathbf{k} \) taken in the right hand sense. But the vector \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \) has the following evaluation

\[
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \emptyset \times \mathbf{k} = \emptyset.
\]

since vectors \( \mathbf{a} \) and \( \mathbf{b} \) are clearly parallel and the \( |\emptyset| = 0 \). This shows that

\[
\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k},
\]

and means that the vector product is NOT associative. Therefore brackets cannot be removed. They MUST remain in both expressions. The expression \( \mathbf{a} \times \mathbf{b} \times \mathbf{c} \) is NOT a defined vector.

2.6.2 Cartesian form of the vector product

The formal geometric definition for the vector product above is not an easy working definition, for how do we find the angles, lengths and of course the normal vector \( \mathbf{n} \) to make the correct evaluation. It is possible to establish other formulae in terms of the coordinates of the vectors relative to the Cartesian orthonormal basis vectors \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) that we have chosen to represent a vector in \( \mathbb{R}^3 \). Using the given definition and the fact that this basis is a set of orthonormal vectors we can establish the following Table 2.2.

\[
\begin{align*}
\mathbf{i} \times \mathbf{i} &= \emptyset, & \mathbf{j} \times \mathbf{i} &= -\mathbf{k}, & \mathbf{k} \times \mathbf{i} &= \mathbf{j}, \\
\mathbf{i} \times \mathbf{j} &= \mathbf{k}, & \mathbf{j} \times \mathbf{j} &= \emptyset, & \mathbf{k} \times \mathbf{j} &= -\mathbf{i}, \\
\mathbf{i} \times \mathbf{k} &= -\mathbf{j}, & \mathbf{j} \times \mathbf{k} &= \mathbf{i}, & \mathbf{k} \times \mathbf{k} &= \emptyset.
\end{align*}
\]

Table 2.2: Vector product basis table

Using this information and some of the properties of the vector product we can find a formula for the vector product of any two vectors in terms of their coordinate representation. Suppose we have two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with coordinate representation in terms of the basis vectors:

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}.
\]

Then proceeding in a like manner to the dot product, remembering to keep the order in each pair of vector products, we find

\[
\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k})
\]
In summary given two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with coordinate representation

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k},
\]

then

\[
\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}.
\]

(2.24)

Again calculations have been written out in explicit detail to show how the various laws and rules apply.

In summary given two vectors \( \mathbf{a} \) and \( \mathbf{b} \) with coordinate representation

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k},
\]

then

\[
\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}.
\]

So the calculation for the vector product is straightforward, given the coordinate representation in terms of the basis vectors. This is called Leibniz’s formula for the vector product.

Hidden away in the formula is the magnitude and direction of the resultant vector product which we can now obtain with no trouble.

The formula is easy enough to remember. Here is a simple way to do this. Consider renaming the basis vectors as

\[
\mathbf{i} = e_1, \quad \mathbf{j} = e_2 \quad \text{and} \quad \mathbf{k} = e_3.
\]

(This is more in line with standard notation for an orthonormal vector in Euclidean space used in texts in Linear Algebra.)
2.6. THE VECTOR (CROSS) PRODUCT

Now each term in the expansion for the vector product can be written as

\[(a_{k+1}b_{k+2} - a_{k+2}b_{k+1})\mathbf{e}_3,\]

where numbers \(k, k + 1, k + 2\), represent permutations of the numbers 1, 2 and 3. For example:

- If \(k = 1\), then the three numbers are 1, 2, 3.
- If \(k = 2\), then the three numbers are 2, 3, 1.
- If \(k = 3\), then the three numbers are 3, 1, 2.

**Alternative determinant definition**

Another popular formula for the vector product requires writing Equation 2.24 in the form

\[\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k.\]

(2.25)

It requires undertaking an illegal calculation of the following determinant expansion with scalar entries:

\[
\begin{vmatrix}
  x_1 & x_2 & x_3 \\
  y_1 & y_2 & y_3 \\
  z_1 & z_2 & z_3 \\
\end{vmatrix}
= \begin{vmatrix}
  y_2 & y_3 & x_1 - y_1 & y_3 & x_2 + y_1 & y_2 & x_3 \\
  z_2 & z_3 & x_1 - z_1 & z_3 & x_2 + z_1 & z_2 & x_3 \\
\end{vmatrix}
= (y_2z_3 - z_2y_3)x_1 - (y_1z_3 - z_1y_3)x_2 + (y_1z_2 - z_1y_2)x_3.
\]

We evaluate as follows

\[
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
\end{vmatrix}
= \begin{vmatrix}
  a_2 & a_3 & a_1 - b_1 & a_3 \\
  b_2 & b_3 & b_1 & a_2 - b_1 & a_3 \\
\end{vmatrix}
= (a_2b_3 - b_2a_3)i - (a_1b_3 - b_1a_3)j + (a_1b_2 - b_1a_2)k
= (a_2b_3 - b_2a_3)i - (a_1b_3 - b_1a_3)j + (a_1b_2 - b_1a_2)k.
\]

This is the same result given in Equation 2.25. Calculations are a blatant illegal evaluation of the determinant, but it works and it is not that hard to remember.

**EXAMPLE 2.13** Find a unit vector perpendicular to \(\mathbf{z} = \mathbf{i} + \mathbf{j} + \mathbf{k}\) and \(\mathbf{y} = \mathbf{i} - 3\mathbf{k}\).

**Solution.** By definition the vector product \(\mathbf{z} \times \mathbf{y}\) is a vector perpendicular to both \(\mathbf{z}\) and \(\mathbf{y}\). We calculate it using the determinant definition.

\[
\mathbf{z} \times \mathbf{y} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  1 & 1 & 1 \\
  1 & 0 & -3 \\
\end{vmatrix} = \begin{vmatrix}
  1 & 1 & 1 \\
  1 & -3 & 1 \\
  1 & 1 & 0 \\
\end{vmatrix} = -3\mathbf{i} + 4\mathbf{j} - \mathbf{k}.
\]
To find a unit vector \( \hat{\mathbf{n}} \) we construct

\[
\hat{\mathbf{n}} = \frac{\mathbf{x} \times \mathbf{y}}{|\mathbf{x} \times \mathbf{y}|} = \frac{\mathbf{x} \times \mathbf{y}}{((-3)^2 + (4)^2 + (-1)^2)^{1/2}} = \frac{1}{\sqrt{26}}(-3\mathbf{i} + 4\mathbf{j} - \mathbf{k}).
\]

Note that \(-\hat{\mathbf{n}}\) could also be given as an answer to this question.

2.6.3 Vector area of a parallelogram

Consider the parallelogram \( OBCA \) shown in Figure 2.23. We observe that the perpendicular distance \( BN \) from \( B \) to \( OA \) is \( BN = |\mathbf{b}| \sin(\theta) \). Note that \( \theta \in [0, \pi] \) and \( \sin(\theta) \) is positive. From the definition of the vector product

\[
|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta) = (\mathbf{OA})(\mathbf{BN}).
\]

This quantity is well recognised as the area of the given parallelogram, being the product of the length of the base and the perpendicular height.

Thus \( \mathbf{a} \times \mathbf{b} \) is a vector perpendicular to parallelogram \( OBCA \) with magnitude equal to the area of the parallelogram itself. We therefore make the following definition.

**DEFINITION 2.11**

The vector area \( \mathbf{A} \) of the parallelogram formed by vectors \( \mathbf{a} \) and \( \mathbf{b} \) is defined to be \( \mathbf{A} = \mathbf{a} \times \mathbf{b} \).
2.6. THE VECTOR (CROSS) PRODUCT

You will find this a most useful concept in latter mathematics, enabling well defined integral definitions to be defined for the surface area of three dimensional surfaces, the representation of the vector moment of force in dynamics and vorticity/rotation in study of fluid motion called fluid mechanics.

EXAMPLE 2.14

(a) Consider a plane parallel to the vectors \( \mathbf{a} = 2\mathbf{i} + \mathbf{j} \) and \( \mathbf{b} = -1\mathbf{i} + \mathbf{k} \) which passes through the point \( C \) having position vector \( \mathbf{c} = 4\mathbf{i} \) relative to a given origin \( O \).

(i) Determine a unit vector perpendicular to this plane.

(ii) Hence determine the Cartesian equation of the plane.

(iii) Show that the point \( E \) with position vector \( \mathbf{e} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} \) lies on the plane.

(b) Let \( D \) be the point with position vector \( \mathbf{d} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \) relative to \( O \).

(i) Find the parametric equation of the line passing through \( D \) which is perpendicular to the plane in (a).

(ii) Hence determine the point where this line intersects the plane.

(iii) What is the perpendicular distance from \( D \) to the plane.

Solution.

(a) (i) The vector \( \mathbf{n} = \mathbf{a} \times \mathbf{b} \) is perpendicular to the plane since it is parallel to both \( \mathbf{a} \) and \( \mathbf{b} \). We find

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & 1 & 0 \\
-1 & 0 & 1
\end{vmatrix} = \begin{vmatrix}
1 & 0 \\
0 & 1 \\
-1 & 1
\end{vmatrix} \mathbf{i} + \begin{vmatrix}
1 & 2 \\
0 & -1 \\
-1 & 0
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
1 & 2 \\
0 & -1 \\
-1 & 0
\end{vmatrix} \mathbf{k} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}.
\]

A unit vector perpendicular to the plane is

\[
\hat{\mathbf{n}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{\mathbf{a} \times \mathbf{b}}{((1)^2 + (-2)^2 + (1)^2)^{1/2}} = \frac{1}{\sqrt{6}}(\mathbf{i} - 2\mathbf{j} + \mathbf{k}).
\]

(ii) Let \( \mathbf{r} \) be the position vector to any point \( P \) on the plane, see Figure 2.24. Clearly then \( \mathbf{C\hat{P}} \) is perpendicular to \( \hat{\mathbf{n}} \) for all such points \( P \). Thus

\[
(\mathbf{r} - \mathbf{c}) \cdot \hat{\mathbf{n}} = 0 \quad \text{or} \quad \mathbf{r} \cdot \hat{\mathbf{n}} = \mathbf{c} \cdot \hat{\mathbf{n}}.
\]
This is the normal equation to the plane. Substituting for point \( P \) with Cartesian coordinates \((x, y, z)\) and \( \vec{r} = xi + yj + zk; \) for \( \hat{n} \) and \( \vec{c} = 4i \), we find
\[
((x - 4)i + yj + zk) \cdot (i - 2j + k)/\sqrt{6} = 0,
\]
evaluates to
\[
x - 2y + z = 4.
\] (2.26)
This is the Cartesian equation of the given plane.

(iii) \( E \) has coordinates \( x = -1, \ y = -2 \) and \( z = 1 \). To determine if \( E \) lies on the plane we must check if the coordinates of the point satisfy the Cartesian equation of the plane. In this case
\[
x - 2y + z = (-1) - 2(-2) + 1 = 4.
\]
Hence \( E \) must lie on the given plane.

(b) (i) The equation of the line which passes through \( D, \ d = -2i + 2j + 2k \) and which is perpendicular to the plane in (a), is given parametrically by
\[
\vec{r}(t) = d + tn \quad \text{where parameter \( t \in \mathbb{R} \)}
\[
= (-2 + t)i + (2 - 2t)j + (2 + t)k.
\]
Note that we do not need to use \( \hat{n} \) as it is but a scalar multiple of \( n \).

(ii) We need to find the value of the parameter \( t \) such that the point with position vector \( \vec{r}(t) \) lies on the plane. At \( P, \ x = -2 + t, \ y = 2 - 2t \) and \( z = 2 + t \). The coordinates must satisfy the equation of the plane for some value of \( t \). We find from Equation 2.26 that
\[
(-2 + t) - 2(2 - 2t) + (2 + t) = 4, \quad \text{or} \quad t = 4/3.
\]
The position vector to the point of intersection $I$, is

$$r(4/3) = (-2\mathbf{i} - 2\mathbf{j} + 10\mathbf{k})/3.$$  

It has coordinates $(-2/3, -2/3, 10/3)$.

(iii) Since the line is perpendicular to the plane, the distance $DI$ is the perpendicular distance from $D$ to the plane. So the perpendicular distance $d_\perp$ equals

$$d_\perp = \left(\left(-2 + 2/3\right)^2 + (2 + 2/3)^2 + (2 - 10/3)^2\right)^{1/2} = 4\sqrt{6}/3.$$

\[\square\]

### 2.6.4 Proof of the left distributive law

Before proving the left distributive law, we introduce an alternative form of definition for the vector product which will allow us to establish the result easily.

Consider any two vectors $\mathbf{a}$ and $\mathbf{b}$ with reference to origin $O$ as shown in Figure 2.25. Let $P$ be any point on the line parallel to $\mathbf{a}$, lying in the plane spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{OP} = \mathbf{b}'$ where $\mathbf{b}'$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, making an angle $\phi$ with the direction of $\mathbf{a}$. We see that

$$|\mathbf{OP}| \sin(\phi) = |\mathbf{OB}| \sin(\theta),$$

![Figure 2.25: Diagram for alternate form of vector product](image-url)
and multiplying both sides of this equation by $|a|$ we have

$$|a||b'| \sin(\phi) = |a||\hat{b}| \sin(\theta),$$

that is $|a \times \hat{b}'| = |a \times \hat{b}|$.

It is also clear that $\hat{a} \times \hat{b}'$ is parallel to $\hat{a} \times \hat{b}$. Since the vectors are equal in magnitude and have the same direction,

$$a \times b' = a \times b.$$

We note that this relation is not unique since $P$ can be taken to be any point on the line through $B$ parallel to $OA$.

For the special case $\phi = \pi/2$, we write $b' = b_{\perp}$, then

$$a \times b = a \times b_{\perp}.$$

The proof of the left distributive law now follows. With this special form of the vector product is suffices us to only show that

$$a \times (b + c)_{\perp} = a \times b_{\perp} + a \times c_{\perp}.$$

Let $OC'D'B'$ be the projection of the parallelogram $OCDB$ on the plane perpendicular to $a$, see Figure 2.26. In the diagram

$$\vec{OD} = b + c, \quad \vec{OB'} = b_{\perp}, \quad \vec{OC'} = c_{\perp}.$$

Clearly $OB'D'C'$ is a parallelogram and

$$\vec{OD'} = b_{\perp} + c_{\perp} = (b + c)_{\perp}.$$

Now

$$a \times b_{\perp}$$

is a vector perpendicular to $a$ and $b_{\perp}$ of magnitude $|a||b_{\perp}|$.

$$a \times c_{\perp}$$

is a vector perpendicular to $a$ and $c_{\perp}$ of magnitude $|a||c_{\perp}|$.

$$a \times (b + c)_{\perp}$$

is a vector perpendicular to $a$ and $(b + c)_{\perp}$ of magnitude $|a||(b + c)_{\perp}|$.

Thus the effect of the vector product of $a \times$ with a vector in the plane perpendicular to $a$ is to ROTATE the parallelogram $OC'D'B'$ through a right angle and MAGNIFY the parallelogram by the factor $|a|$. This means that $a \times (b + c)_{\perp}$ will be the diagonal of the new parallelogram.

Hence from the parallelogram or triangle law of addition, we have the result

$$a \times (b + c)_{\perp} = a \times b_{\perp} + a \times c_{\perp},$$

which implies the left distributive law of vector product over vector addition.
2.7 Products of more than two vectors

2.7.1 The scalar triple product

Given three vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) it is possible to calculate the scalar quantity \( \vec{a} \cdot (\vec{b} \times \vec{c}) \). This quantity is called the scalar triple product of the vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \).

Note the result of the calculation is indeed a scalar and remember that \((\vec{a} \cdot \vec{b}) \times \vec{c}\) is not a defined quantity. Why? Because the result of the calculation in the brackets gives a scalar and you cannot take the vector product of a scalar and a vector.

Let us take an origin \( O \) in \( \mathbb{R}^3 \), and let points \( A, B, \) and \( C \) be the points with position vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) relative to the origin. Construct the parallelopiped with \( OA, OB \) and \( OC \) as concurrent edges, see Figure 2.27. In the diagram it is assumed that vector \( \vec{a} \) makes an acute angle with the vector \( \vec{b} \times \vec{c} \). In this case vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \) form what is called a right hand system.

Set \( \vec{n} = \vec{b} \times \vec{c} \), then

\[
\hat{n} = \frac{\vec{b} \times \vec{c}}{|\vec{b} \times \vec{c}|}, \quad \text{and} \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \hat{n}|\vec{b} \times \vec{c}|.
\]
Now we know that $|\vec{b} \times \vec{c}|$ is the area of the base of the parallelogram $OBDC$, and $\vec{a} \cdot \hat{n}$ is the perpendicular height $AN$ of the solid parallelopiped.

Hence the volume $V$ of the parallelopiped is simply

$$V = \vec{a} \cdot (\vec{b} \times \vec{c}).$$

If $\vec{a}$ makes an obtuse angle with $\vec{b} \times \vec{c}$ then the product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = -V.$$

In this case the vectors in order $\vec{a}, \vec{b}$ and $\vec{c}$ form a left hand system of three vectors.

We conclude that the volume $V$ of the parallelopiped formed by any set of three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is given by the general formula:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$  \hspace{1cm} (2.27)

Since 6 identical tetrahedrons formed by these three vectors fit into this parallelopiped, the volume $V_T$ of the tetrahedron $OACB$ is given by

$$V_T = (1/6)|\vec{a} \cdot (\vec{b} \times \vec{c})|.$$  \hspace{1cm} (2.28)
Coordinate representation of the scalar triple product

Suppose we have given origin \( O \) and an orthonormal basis \( \{ \mathbf{i}, \mathbf{j}, \mathbf{k} \} \). Let the vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) have coordinate representation

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}, \quad \text{and} \quad \mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k}.
\]

Using properties of the scalar product and vector product previously derived we find that

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \left| \begin{array}{ccc}
    i & j & k \\
    b_1 & b_2 & b_3 \\
    c_1 & c_2 & c_3 \\
\end{array} \right| = b_2 c_3 - b_3 c_2.
\]

Exact details in these calculations are left to the reader to verify.

**Properties of the scalar triple product**

**STP1** Viewing the parallelopiped we have constructed from a different position, it is easy to observe that

\[
V = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).
\]

The value of the scalar triple product remains the same when vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) are cyclically permuted. If the vectors are not in cyclic order as in the above it is clear that the values obtained will be equal to \( -V \), that is,

\[
-V = \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = \mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}).
\]

**STP2** It can be observed from property STP1 that the two vector binary operators \( \times \) and \( \cdot \) may be interchanged without change in value to the scalar triple product. For example, check that

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.
\]
remembering of course to correctly place the brackets around the vector product formed.

**STP3** Since we know that the vector product of any two parallel vectors is simply the zero vector, we conclude that if any two vectors are parallel in the scalar triple product \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \), then the scalar triple product must be zero, that is, \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \).

**STP4** The following results should be obvious

(i) If \( \mathbf{a} \) lies in the plane formed by \( \mathbf{b} \) and \( \mathbf{c} \), then \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \). (Clearly the parallelopiped has zero volume.)

(ii) If \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \), then the volume of the parallelopiped formed by the three vectors must be zero. If the vectors are all non-zero vectors then the only conclusion we can draw is that \( \mathbf{a} \) must lie in the plane spanned by \( \mathbf{b} \) and \( \mathbf{c} \).

In summary:

The four points \( O, A, B \) and \( C \) are coplanar, that is, vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are coplanar, IF AND ONLY IF, \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = 0 \).

**STP5** Following from property STP4, we can say that a necessary and sufficient condition for the non-zero vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) to be linearly independent is that \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \neq 0 \). Another way of writing this is as follows:

Three non-zero vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) are linearly independent IF AND ONLY IF \( \mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \neq 0 \).

### 2.7.2 The vector triple product

Given three vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \) it is also possible to calculate the vector quantity \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \). This quantity is called the *vector triple product* of the vectors \( \mathbf{a}, \mathbf{b} \) and \( \mathbf{c} \). This quantity is a vector and it is important that the brackets are kept in its description and evaluation. Why? Because we know that the vector product is not associative.

The vector triple product \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) and of course the other vector triple product \( (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \) each admit an easy expansion:

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{2.32}
\]

\[
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a} \tag{2.33}
\]

From these equations we observe
2.7. PRODUCTS OF MORE THAN TWO VECTORS

The vector \( \vec{a} \times (\vec{b} \times \vec{c}) \) lies in the plane spanned by the vectors \( \vec{b} \) and \( \vec{c} \) as it is a linear combination of these two vectors.

The vector \( (\vec{a} \times \vec{b}) \times \vec{c} \) lies in the plane spanned by the vectors \( \vec{b} \) and \( \vec{a} \) as it is a linear combination of these two vectors.

Assuming that the three vectors are non-zero and distinct, that is they are all different, each triple vector product of these three vectors lies in a different plane. We shall verify the expansion given in Equation 2.33.

ILLUSTRATION 2.3
For given vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \), verify that the vector triple product expansion
\[
(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}.
\]

Solution.

By the definition of the vector product, the vector \( (\vec{a} \times \vec{b}) \times \vec{c} \) must certainly be perpendicular to \( \vec{a} \times \vec{b} \) and so it must lie in the plane of the two vectors \( \vec{a} \) and \( \vec{b} \).

Hence we must be able to write \( (\vec{a} \times \vec{b}) \times \vec{c} \) as a linear combination of \( \vec{a} \) and \( \vec{b} \). So there exist scalars \( \alpha \) and \( \beta \) for which
\[
(\vec{a} \times \vec{b}) \times \vec{c} = \alpha \vec{a} + \beta \vec{b}.
\]

We need to now precisely determine the two scalars.

Since \( (\vec{a} \times \vec{b}) \times \vec{c} \) is clearly perpendicular also to \( \vec{c} \), that is
\[
((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{c} = (\alpha \vec{a} + \beta \vec{b}) \cdot \vec{c} = 0,
\]

from which we obtain
\[
\alpha (\vec{a} \cdot \vec{c}) + \beta (\vec{b} \cdot \vec{c}) = 0. \tag{2.34}
\]

This equation defines the ratio \( \alpha/\beta \) or \( \beta/\alpha \). Let \( \alpha = -\lambda (\vec{b} \cdot \vec{c}) \) and \( \beta = \lambda (\vec{a} \cdot \vec{c}) \), then Equation 2.34 is satisfied for the single unknown \( \lambda \). Further
\[
(\vec{a} \times \vec{b}) \times \vec{c} = \lambda (-(\vec{b} \cdot \vec{c})\vec{a} + (\vec{a} \cdot \vec{c})\vec{b}). \tag{2.35}
\]

To establish the desired expansion we now have to show that \( \lambda = 1 \), no matter the definition of the three vectors \( \vec{a}, \vec{b} \) and \( \vec{c} \).

Special case. We first consider a special case when \( \vec{c} = \vec{a} \) that will enable us to prove the general result. In this case
\[
(\vec{a} \times \vec{b}) \times \vec{a} = \lambda ((\vec{a} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{a}). \tag{2.36}
\]
Take the dot product of both sides of Equation 2.36 with \( \vec{b} \), we obtain:

\[
[(a \times b) \times a] \cdot \vec{b} = \lambda((a \cdot a)(\vec{b} \cdot \vec{b}) - (\vec{b} \cdot a)(a \cdot b)).
\] (2.37)

Consider the evaluation of the left hand side of Equation 2.37.

\[
[(a \times b) \times a] \cdot \vec{b} = [\vec{d} \times a] \cdot \vec{b} \quad \text{where} \quad \vec{d} = a \times \vec{b}
\]

\[
= \vec{d} \cdot (a \times \vec{b}) \quad \text{by property STP2}
\]

\[
= (a \times \vec{b}) \cdot (a \times \vec{b}) = |a \times \vec{b}|^2
\]

\[
= |a|^2 |\vec{b}|^2 \sin^2(\theta) \quad \text{where} \theta \text{ is the angle between} \ a \ \text{and} \ \vec{b}
\]

\[
= |a|^2 |\vec{b}|^2 (1 - \cos^2(\theta))
\]

\[
= |a|^2 |\vec{b}|^2 - |a|^2 |\vec{b}|^2 \cos^2(\theta)
\]

\[
= |a|^2 |\vec{b}|^2 - |\vec{b}|^2 |a| \cos(\theta))^2
\]

\[
= (a \cdot a)(\vec{b} \cdot \vec{b}) - (a \cdot \vec{b})^2.
\]

Comparing this evaluation with the right hand side of Equation 2.37, for arbitrary \( \vec{a} \) and \( \vec{b} \), then \( \lambda = 1 \).

**General case** We now prove \( \lambda = 1 \) for the general case.

Consider Equation 2.35. Taking the dot product of both sides of this equation with the vector \( \vec{a} \) we obtain

\[
[(a \times b) \times c] \cdot \vec{a} = \lambda((a \cdot c)(\vec{b} \cdot \vec{a}) - (\vec{b} \cdot c)(a \cdot \vec{a})).
\] (2.38)

Now using property STP1, the left handside of this Equation 2.38 equals

\[
[(a \times b) \times c] \cdot \vec{a} = [\vec{d} \times c] \cdot \vec{a} = -[\vec{d} \times a] \cdot \vec{c} = -[(a \times b) \times a] \cdot \vec{c},
\]

where \( \vec{d} = a \times \vec{b} \). Inside the last term of this equation is \( (a \times b) \times a \) which we have already examined in the special case above. So using the fact that

\[
(a \times b) \times a = (a \cdot a)\vec{b} - (b \cdot a)\vec{a},
\]

we find that

\[
[(a \times b) \times c] \cdot \vec{a} = -[(a \times b) \times a] \cdot \vec{c} = -[(a \cdot a)\vec{b} - (b \cdot a)\vec{a}] \cdot \vec{c}
\]

\[
= (b \cdot a)(a \cdot \vec{c}) - (a \cdot a)(b \cdot \vec{c}).
\]

Comparing now this expression with the right hand of Equation 2.38, we conclude that since \( \vec{a}, \vec{b} \) and \( \vec{c} \) were arbitrary vectors, \( \lambda = 1 \). This completes the proof.
REMARK 2.5
Notice this proof of the expansion of $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ was independent of the nature of any coordinate system that we may choose to represent the vectors.

Indeed there is no nice neat Cartesian formula for the evaluation of the vector triple product and there is no nice associated geometry as there was with the scalar triple product. It is important however to know the result for expansion of the vector triple product when it occurs in calculations. For it is much easier to evaluate using the dot products, than to handle two determinant calculations in the evaluations of two vector products.

2.8 Further exercises

In this section some exercises are given in no particular order. Please also refer to the tutorial sheets for other exercises.

EXERCISE 2.8
Prove that given any two vectors $\mathbf{a}$ and $\mathbf{b}$

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|.$$  

This is called the triangle inequality.

EXERCISE 2.9
Find a unit vector perpendicular to each of the vectors $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, and the angle between these vectors.

EXERCISE 2.10
Show that the points $(1, -1, 3)$ and $(3, 3, 3)$ are equidistant from the plane

$$5x + 2y - 7z = -9,$$

and on opposite sides of it.
EXERCISE 2.11
Show that the plane through the point $(2, -4, 5)$ perpendicular to the line of intersection of the planes

$$2x + 3y - 4z = 1, \text{ and } 3x + y - 2z = 2,$$

has Cartesian equation

$$2x + 8y + 7z = 7.$$  

EXERCISE 2.12
Find the distance $d$ from the point $(4, -2, 3)$ to the straight line through the point $(3, 1, -4)$ and parallel to the vector $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. (Answer: $d \approx 6.06$)

EXERCISE 2.13
Point $P$, $Q$, $R$ and $S$ have position vectors $\mathbf{p} = \mathbf{i} - \mathbf{k}$, $\mathbf{q} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{r} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ respectively. Show that the projection of $PQ$ on $RS$ is equal to that of $RS$ on $PQ$, each being $-4/3$. Also show that the cosine of their inclination is $-4/9$.

EXERCISE 2.14
Find the intersection of the line joining the points $(1, -2, -1)$ and $(2, 3, 1)$ with the plane through the points $(2, 1, -3)$, $(4, -1, 2)$ and $(3, 0, 1)$. (Answer: $(5/3, 4/3, 1/3)$)

EXERCISE 2.15
Prove by vector methods that in any parallelogram the sum of the squares on the diagonals is twice the sum of the squares on two adjacent sides.

EXERCISE 2.16
Show by vector methods that the perpendiculars from the vertices of a triangle to the opposite sides are concurrent, that is they have a common point of intersection.

EXERCISE 2.17
Show that the acute angle between two diagonals of a cube has cosine $1/3$.

EXERCISE 2.18
Show by vector methods that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.
EXERCISE 2.19
Show that the vector area $\vec{A}$ of the triangle, whose vertices $A$, $B$ and $C$ are given by the position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ relative to a given origin $O$, is given by

$$\vec{A} = (1/2)(\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}).$$

EXERCISE 2.20
Find the volume of the tetrahedron whose vertices are the points $A(2, -1, -3)$, $B(4, 1, 3)$, $C(3, 2, -1)$ and $D(1, 4, 2)$. 